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Deterministic coherence resonance in systems with on-off intermittency and delayed feedback

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Abstract. Coherence resonance consists in the increase of regularity of an output signal of a nonlinear device for non-zero intensity of input noise. This phenomenon occurs, e.g., in stochastic systems with delayed feedback in which external noise amplifies the periodic component of the output signal with the period equal to the delay time. In this contribution it is shown that in chaotic systems with delayed feedback deterministic (noise-free) coherence resonance can occur, which consists in the maximization of the periodic component of the output signal in the absence of stochastic noise, due to the changes in the internal chaotic dynamics of the system as the control parameter is varied. This phenomenon is observed in systems with on-off intermittency and attractor bubbling, including generic maps and systems of diffusively coupled chaotic oscillators at the edge of synchronization. The occurrence of deterministic coherence resonance for the optimum value of the control parameter (e.g., of the coupling strength between synchronized oscillators) is characterized by the appearance of a series of maxima at the multiples of the delay time in the probability distribution of the laminar phase lengths, superimposed on the power-law trend typical of on-off intermittency, and by the presence of a strong maximum in the power spectrum density of the output signal.
Keywords: on-off intermittency, coherence resonance, delayed feedback.

1 Introduction

On-off intermittency (OOI) is a sort of extreme bursting which occurs in systems posessing a chaotic attractor within an invariant manifold whose dimension is less than that of the phase space [1,2]. As a control parameter crosses a certain threshold this attractor undergoes a supercritical blowout bifurcation [3] and loses transverse stability, and a new attractor is formed which encompasses that contained within the invariant manifold. Just above the blowout the phase trajectory stays for long times close to the invariant manifold and occasionally departs from it; if the distance from the invariant manifold is an observable, this results in a sequence of laminar phases and bursts. The distribution of laminar phase lengths τ obeys a power scaling law $P(\tau) \propto \tau^{-3/2}$ [1]. In the presence of additive noise chaotic bursting occurs below the blowout

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bifurcation threshold; this phenomenon is known as attractor bubbling [2,4]. OOI and attractor bubbling were observed in systems as diverse as model maps with time-dependent control parameter [1], chaotic synchronization [5], spin-wave chaos [6], microscopic models of financial markets [7], etc.

The role of delayed feedback is important in many systems, e.g. optical resonators, chemical reactions and physiology [8] or chaos control [9,10]. In this paper the influence of delayed feedback on OOI is studied using generic one-dimensional maps with a time-dependent control parameter and synchronized oscillators. It is shown that addition of delayed feedback changes the threshold for the blowout bifurcation and can influence the character of the intermittent bursting: For optimum choice of the control parameter a strong periodic component in the time series above the blowout occurs, with the period equal to the delay time. This is an example of coherence resonance (CR) [11-18], a phenomenon related to the well-known stochastic resonance (SR) [19]. CR manifests itself as the peak of regularity of the output signal of certain nonlinear stochastic systems for optimum intensity of the input noise and without any external periodic stimulation. In particular, CR was observed in systems with delayed fedback, including bistable [16] and excitable [17] ones and simple threshold crossing detectors [18]. Since in the models under consideration the role of external noise is played by the internal chaotic dynamics within the invariant manifold, the observed phenomenon is deterministic CR [20], a counterpart of the noise-free (deterministic) SR [21].

2 Modeling with a Logistic Map with a time-dependent control parameter and delayed feedback

As a basic model let us consider the logistic map with the time-dependent control parameter and delayed feedback

$$y_{n+1} = (1-K)\, a\zeta_n y_n\, (1-y_n) + K y_{n-k}, \tag{1}$$

where $0 < K < 1$ is the amplitude of the feedback term and $\zeta_n \in (0,1)$ denotes any chaotic process constrained to the unit interval. The map in Eq. (1) has the invariant manifold $y_n = 0$ with the chaotic attractor $(\zeta_n \in (0,1), y_n = 0)$ within it. For $a > a_c$ the variable y_n exhibits intermittent bursts, where a_c is the blowout bifurcation threshold dependent on ζ_n. For $K = 0$ Eq. (1) is the generic model for OOI [1]. The qualitative properties of OOI are independent of the details of the dynamics within the invariant manifold provided that the correlation time of the process ζ_n is negligible in comparison with the mean time between bursts, which is true just above the threshold for the blowout bifurcation; hence, ζ_n can be approximated by white noise ξ_n uniformly distributed on $(0,1)$ [1]. It should be also noted that Eq. (1) with the control parameter constant in time, i.e., with $\zeta_n \equiv 1$, (the logistic map with delayed feedback) can serve as a model for chaos control [10].

For $y_n \approx 0$ the dynamics transverse to the invariant manifold is well approximated by a linearization of Eq. (1),

$$y_{n+1} \approx (1-K)\, a\zeta_n y_n + K y_{n-k}. \tag{2}$$

Introducing new variables in the direction transverse to the invariant manifold, $y_n^{(1)} = y_n$, $y_n^{(2)} = y_{n-k}, \ldots, y_n^{(j)} = y_{n-k+j-2}, \ldots, y_n^{(k+1)} = y_{n-1}$ [10] Eq. (2) can be written as a linear transformation

$$\mathbf{y}_{n+1} = \hat{M}_n \mathbf{y}_n, \tag{3}$$

where $\mathbf{y}_n = \left(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k+1)}\right)^T$ (thus, $\mathbf{y}_n = 0$ is the invariant manifold), and

$$\hat{M}_n = \begin{pmatrix} (1-K)\,a\zeta_n & K & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{4}$$

The transverse stability of the attractor within the invariant manifold is controlled by the transverse Lyapunov exponent λ_T [1-3],

$$\lambda_T = \lim_{N\to\infty} \frac{1}{N} \ln \frac{\left|\left|\hat{M}_{N-1} \ldots \hat{M}_2 \hat{M}_1 \mathbf{y}_0\right|\right|}{||\mathbf{y}_0||}, \tag{5}$$

where $\mathbf{y}_0$ is an arbitray initial vector transverse to the invariant manifold (in simulations, $\mathbf{y}_0$ is assumed as a random vector of unit length). The exponent λ_T increases with a from negative to positive values and crosses zero at the threshold for the blowout bifurcation $a = a_c$, corresponding to the onset of OOI.

The dependence of a_c on K for the map (1) with $\zeta_n = \xi_n$ and various k is shown in Fig. 1(a). The value of a_c weakly depends on k and monotonically decreases to $a_c = 2.0$ for $K \to 1$. Typical time series y_n for a just above a_c is shown in Fig. 1(b). For increasing K the character of the time series changes from intermittent bursts with high amplitude typical of OOI to frequent bursts with small amplitude. There is also a gap between the minimum value of y_n and the invariant manifold $y_n = 0$. Thus the effect of the delayed feedback on the generic model for OOI resembles that of additive noise which prevents the phase trajectory from approaching closely the invariant manifold and lowers the threshold for the occurrence of bursting, leading to attractor bubbling [2,4]. This is not surprising since the additive noise enters Eq. (1) in the same way as the feedback term; moreover, especially for long k, due to decreasing correlation, the feedback term can be treated as a sort of deterministic noise.

For $K > 0$ the distribution of laminar phase lengths $P(\tau)$ for a just above a_c exhibits a series of maxima at the values of τ equal to k and its multiples (Fig. 1(c)) superimposed on a power-law trend typical of OOI. Let us define the output signal as $Z_n = 0$ if y_n is in the laminar phase and $Z_n = 1$ if y_n is in the burst phase (such discretization is typical in the study of systems with SR). Then, a broad peak centered at the frequency $2\pi/k$ appears in the power spectrum density (PSD) of Z_n (Fig. 1(d)). Both absolute and relative (with respect to the mean value of the PSD on the interval $(\pi/k, 3\pi/k)$) height of this peak exhibit maximum as functions of a (Fig. 1(e)); these quantities

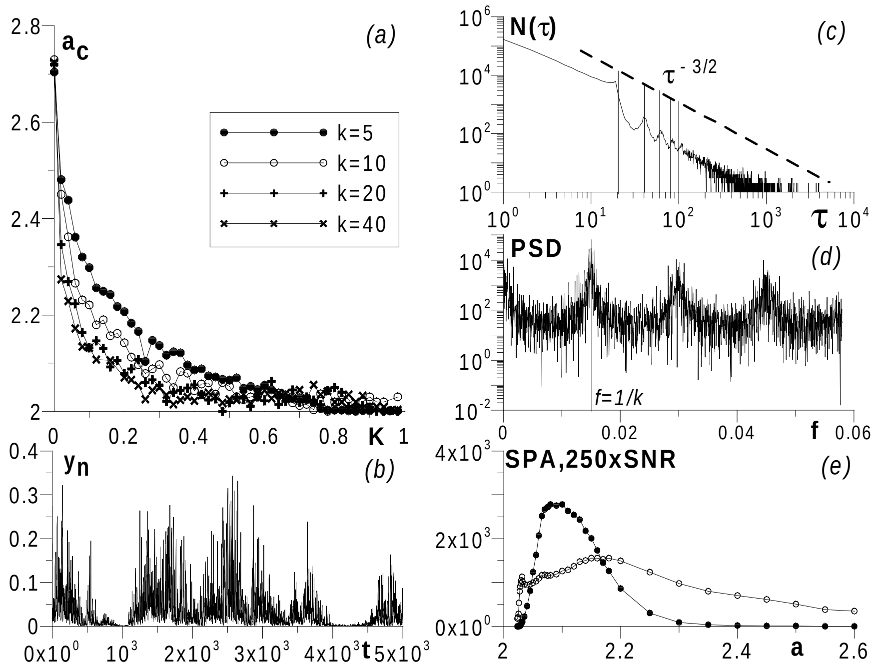

Fig. 1. For the map given by Eq. (1) with $\zeta_n = \xi_n$: (a) intermittency threshold a_c vs. K for various k (see legend); (b) time series $y_n(t)$ for $k = 20$, $K = 0.2$, $a = 2.2$ (just above a_c), the initial condition is $y_0 \in (0, 1)$, where y_0 is a random number, and $y_{-1} = y_{-2} = \ldots y_{-k+1} = 0$; (c) histogram of the number of laminar phases $N(\tau)$ of duration τ for $k = 20$, $K = 0.3$, $a = 2.1$, y_n was assumed to be in the burst phase ($Z_n = 1$) if $y_n > 0.01$, vertical lines are drawn at multiples of k; (d) PSD from the time series Z_n for $k = 64$, $K = 0.3$, $a = 2.1$; (e) SPA (dots) and 250 SNR (circles) vs. a for $k = 64$, $K = 0.3$.

correspond to the spectral power amplification (SPA) and signal-to-noise ratio (SNR) used in the studies of SR, respectively. The height of these maxima increases, their width decreases and their location approaches $a = a_c$ as $K \to 1$ since then the feedback term becomes dominant in Eq. (1) and the signal Z_n is almost periodic for a just above a_c.

These results show that CR occurs in the map (1) as the control parameter is increased above the threshold for the blowout bifurcation. In fact, systems with OOI resemble excitable ones, in particular just above the intermittency threshold when the bursts are short in comparison with the quiescent laminar phases. Thus, CR in the map (1) resembles that observed in excitable systems and threshold-crossing detectors with delayed feedback and external noise [17,18], e.g., the multiple maxima in the histogram of laminar phase lengths in Fig. 1(c) correspond to those found in the histograms of inter-spike intervals in excitable systems with CR [12]. However, CR in the map (1) appears due to changes of the internal dynamics within the invariant manifold as the control parameter is varied rather than under the influence of external noise. Thus, this phenomenon belongs to the class of deterministic CR as in Ref. [20].

3 Modeling with a system of two diffusively coupled chaotic Rössler oscillators

Similar phenomena were observed in a system of two diffusively coupled chaotic Rössler oscillators,

$$\begin{aligned}
\dot{x}_1 &= -\left(y_1 + z_1\right) \\
\dot{y}_1 &= x_1 + a y_1 + k\left(y_2 - y_1\right) + K s(\tau) \\
\dot{z}_1 &= b + z_1\left(x_1 - c\right) \\
\dot{x}_2 &= -\left(y_2 + z_2\right) \\
\dot{y}_2 &= x_2 + a y_2 + k\left(y_1 - y_2\right) - K s(\tau) \\
\dot{z}_2 &= \left(b + \delta b\right) + z_2\left(x_2 - c\right),
\end{aligned} \tag{6}$$

where $a = 0.2$, $b = 0.2$, $c = 11$, k is the strength of the diffusive coupling, $s(\tau) = y_2(t-\tau) - y_1(t-\tau) = \Delta y\,(t-\tau)$ provides delayed feedback with delay τ and amplitude K, and small $\delta b \neq 0$ can be added to model the mismatch of parameters in an experimental system. For $K = 0$ and $\delta b = 0$ the oscillators are identically synchronized for $k > k_c \approx 0.12$ and there is a chaotic attractor within the invariant synchronization manifold $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$. For $k < k_c$ synchronization is lost (i.e., the invariant manifold loses transverse stability) and $\Delta y(t) = y_2(t) - y_1(t)$ exhibits chaotic bursts typical of OOI; thus, k is the control parameter for the supercritical blowout bifurcation. For $\delta b \neq 0$ bursts occur already for $k > k_c$ due to attractor bubbling. Similarily, the delayed feedback $Ks(\tau)$ with $K > 0$ also forces the trajectory to leave the invariant mainfold, as in Eq. (1), and causes the onset of intermittent bursts for $k > k_c$.

Typical time series $\Delta y(t)$ exhibiting OOI are shown in Fig. 2(a). If, again, the output signal is defined as $Z(t) = 0$ if $\Delta y(t)$ is in the laminar phase and $Z(t) = 1$ if $\Delta y(t)$ is in the burst phase, a broad peak centered at the frequency $2\pi/\tau$ appears in the PSD of $Z(t)$ for a range of k below and just above k_c (Fig. 2(b)). The height of this peak (SPA) exhibits maximum as a function of k, both for $\delta b = 0$ and $\delta b > 0$ (Fig. 2(c)); in the latter case only the range of the control parameter where the bursts are observed is slightly broadened toward higher values. This demonstrates that deterministic CR occurs in the system given by Eq. (6) and the output signal exhibits maximum regularity for optimum value of the parameter k which controls the internal dynamics within the invariant synchronization manifold. The maximum of the SNR vs. k is not clearly visible (Fig. 2(d)): evaluating PSD from much longer time series would probably lead to smoother curves of the SNR. Hence, the results of numerical simulations suggest that deterministic CR can be observed experimantally in systems of coupled chaotic oscillators at the edge of identical synchronization.

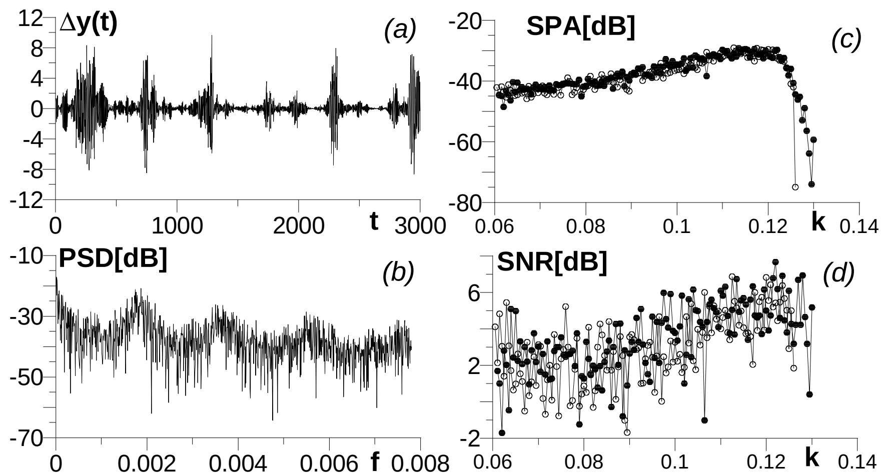

Fig. 2. For the system of diffusively coupled Rössler oscillators given by Eq. (6) with $\tau = 512$, $K = 0.05$, (a) time series $\Delta y(t)$ for $k = 0.12$, $\delta b = 10^{-4}$; (b) PSD from the time series $Z(t)$ for $k = 0.12$, $\delta b = 10^{-4}$, $\Delta y(t)$ was assumed to be in the burst phase ($Z(t) = 1$) if $\Delta y(t) > 0.1$; (c) SPA and (d) SNR vs. k for $\delta b = 0$ (circles) and $\delta b = 10^{-4}$ (dots)

4 Summary

To summarize, the influence of delayed feedback on OOI was studied using generic maps with the time-dependent control parameter and synchronized chaotic oscillators. It was found that delayed feedback can decrease the threshold for the blowout bifurcation. Deterministic CR was observed in systems under consideration, characterized by the appearance of a series of maxima at the multiples of the delay time in the probability distribution of the laminar phase lengths, superimposed on the power-law trend typical of OOI, and by the presence of a strong periodic component in the intermittent time series, with period equal to the delay time. The strength of this component exhibits maximum as the control parameter is varied, due to the changes of the internal dynamics of the system within the invariant manifold.

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