Spatiotemporal Chaos due to Spiral Waves Core Expansion

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Abstract: In the framework of the Fitzhugh Nagumo kinetics and the oscillatory recovery in excitable media, we present a new type of meandering of the spiral waves, which leads to spiral break up and spatiotemporal chaos. The tip of the spiral follows an outward spiral-like trajectory and the spiral core expands in time. This type of destabilization of simple rotation is attributed to the effects of curvature and the wave-fronts interactions in the case of oscillatory damped recovery to the rest state. This model offers a new route to and caricature for cardiac fibrillation.

Keywords: Spiral break up, spatiotemporal chaos.

1. Introduction

Rotating spiral waves are ubiquitous in excitable media. They have been observed in chemical reactive solutions [1, 2], in slime-mold aggregates [3] and most importantly in cardiac muscle [4].Such wave patterns have been studied using reaction-diffusion equations models. For some values of the system control parameters, they undergo simple rigid rotation around a circular core. However, as the control parameter is varied, the spiral tip deviates from circular trajectories [5-11]. This non-steady rotation is known as meandering and it has been observed essentially in chemical systems such as in the Belouzov-Zhabotinsky (BZ) reaction [12]. Experiments with this reaction have also demonstrated spiral breakup [13, 14]. This later is of interest in cardiology since it is the prelude to cardiac fibrillation, the commonest cause of sudden cardiac death [15, 16], and has been observed in models that show wave trains spatiotemporal instabilities [17-18]. It is characterized by spatiotemporally chaotic or irregular wave patterns in excitable media and remains a challenging problem in nonlinear science.

We present in this paper a new type of meandering leading to spiral breakup and offering a new route to spatiotemporal irregularity or chaos in excitable media. Spiral core expansion occurs here as the spiral free end or tip follows an outward motion along a path that looks itself like a spiral. This core expansion was previously expected by the theory of non-local effects [6, 9, 10], and was

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attributed to effect of curvature on the velocity of propagation coupled to the effects of the interaction of successive wave-fronts due to refractoriness. The dependence of the normal velocity of propagation on curvature is given by $v = v_0 - kD$, where k is the local curvature, D is the diffusion coefficient and v_0 is the plane wave velocity of propagation [11]. Due to this velocity gradient, small wavelength perturbations on the segments away from the tip would decay, which would stabilize wave propagation away from the tip and maintains the rotational motion of the spiral. On the contrary, perturbations straightening a small segment containing the tip would reduce curvature, and consequently the normal velocity of wave propagation is enhanced as the gradient of the normal velocity becomes weaker. This means that the tip would have a less tendency to curl but it tends to advance further. Therefore, further straightening of this segment containing the tip is expected. Thus, the spiral tip undergoes an outward forward motion instead of simple rigid rotation. If the recovery is non-oscillatory but monotonic, this destabilizing effect of curvature would be counteracted by the repulsive wave-front interaction due to the refractory period imposed on the medium after the passage of the preceding wave. In that case, circular rigid rotation would be sustained.

This outward motion of the tip along a spiraling trajectory was predicted by Ehud Meron in his theory of non-local effects [6, 10]. He proposed an approximate spiral wave solution of the reaction diffusion system in the form of a superposition of solitary wave-fronts parallel to each other, and then derived an evolution equation using a singular perturbation approach. The numerical solution of this equation, for the case of an oscillatory recovering excitable medium, was a spiral wave whose core expands in time and whose tip moves itself along a spiraling path. However, no observation of this type of spiral wave meandering and core expansion was obtained by Meron in reaction diffusion systems.

2. The Model

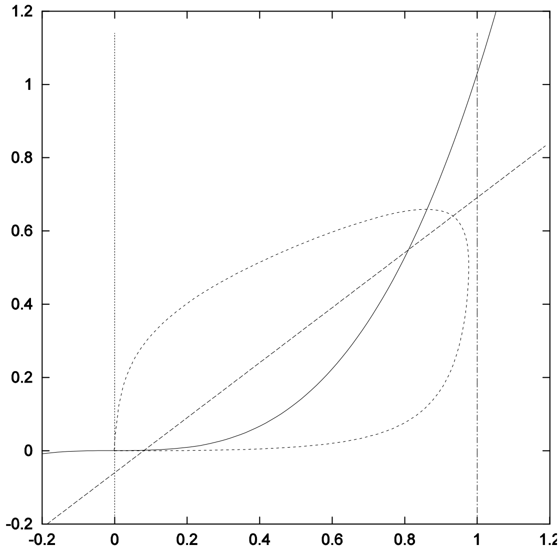
Here, we present a new model showing for the first time this predicted core expansion. We use a modified Barkley's model [19, 20] given by:

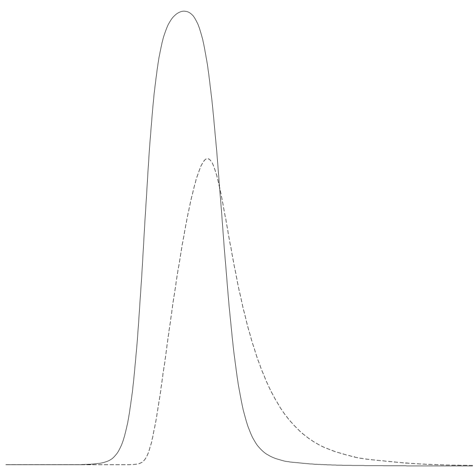
$$\partial u / \partial t = \frac{1}{\pounds} u(1-u)[u - ((b+v)/a)] + \nabla^2 u \,,$$

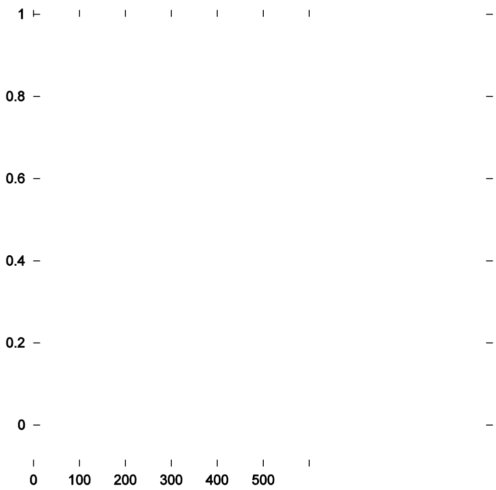
$$\partial v / \partial t = u^3 - v \,, \qquad (1)$$

where u and v are the excitation and recovery variables respectively. The parameter b determines the excitation threshold. The inverse of $\pounds$, characterizing the abruptness of excitation, determines the recovery time. In the standard Barkley's model where the local kinetics in the second equation is given by $(u - v)$, propagation cannot be maintained upon increasing $\pounds$. Here propagation is maintained due to the delay in the production of v.

Numerical simulations were performed on square grids using the explicit Euler integration method with a 9-point neighborhood of the Laplacian and no-flux boundary conditions. The space and time steps are respectively dx = 0.51 and dt = 0.052.



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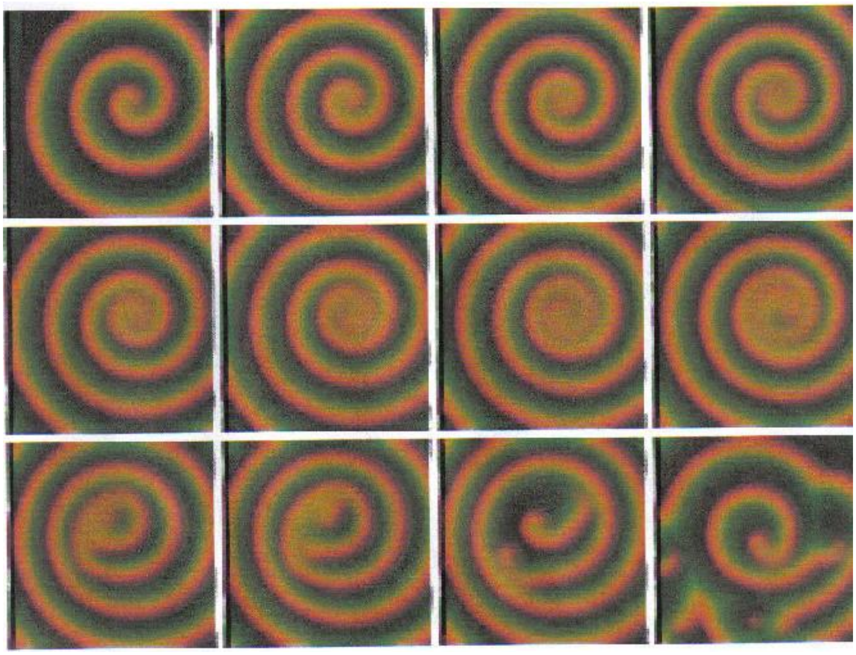

Fig. 2. Snapshots showing the core expansion with dt = 0.052, dx = 0.51, L = 48, grid size: 95. (Time intervals between snapshots are not equal).

prediction of the theory of non-local effects by Meron in the case of oscillatory recovery in excitable media. The type of recovery actually depends on the value of the control parameter £ . For low values of £ , perturbations near the spiral core are quenched by repulsive wave-fronts interactions in the monotonically recovering medium. If the value of £ is increased and for appropriate values for the other parameters, the tip undergoes this interesting outward motion along a path that looks like a spiral while the core grows in size due to oscillatory recovery to the resting state. The distance between the tip segment and the one ahead of it is determined by one of the maxima of the oscillatory tail. This corresponds to one of the minima of the excitation threshold.

In Fig. 3(a), the distance between the points denoting the tip positions increases as the tip moves outward implying that the tip motion is accelerated. In Fig.2, the spiral core expands until spiral breakup occurs. This happens because the spiral period changes as the spiral drifts and meanders, until at some point within the excitable medium, it reaches the minimum period needed for plane wave propagation. This change in the spiral period as the tip moves outward and forward is due to Doppler shift since the core is seen as the source of waves. This means that conduction would be blocked since the spiral rotates more rapidly than plane waves can propagate when the spiral period and the minimum period compatible with plane wave propagation merge for this critical value of

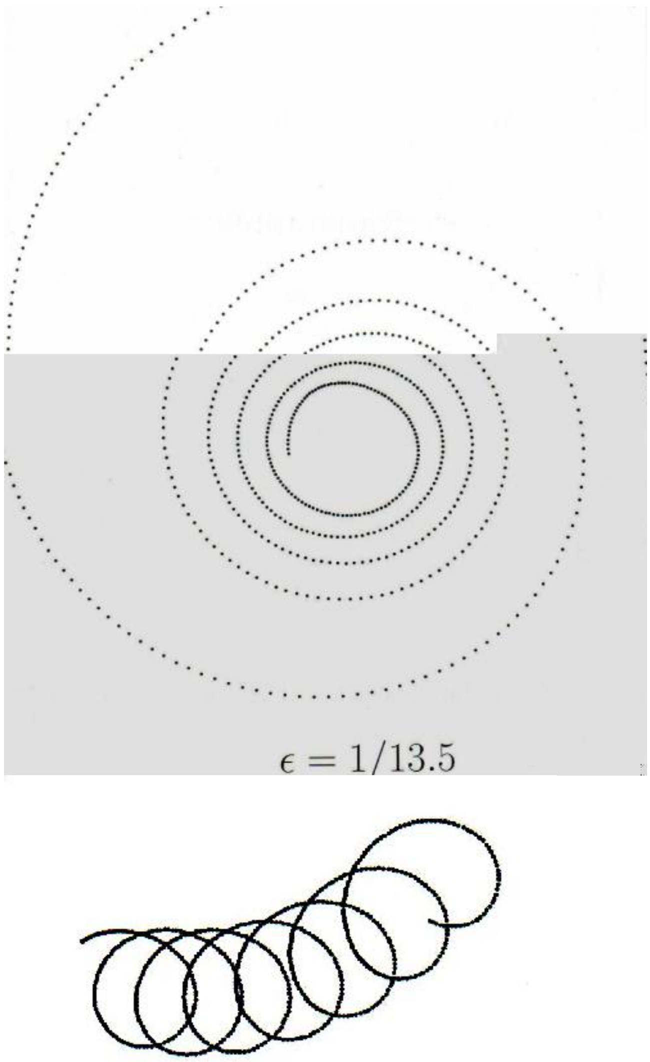

Fig. 3. Trajectory of the spiral tip defined as the intersection of the isolines u = v = 0.5, and following an outward spiral trajectory in (a) and meandering in (b). Parameters are the same as in Fig. 1 in (a), and in (b) $£^{-1} = 18.0$.

£ [22]. The spiral becomes unstable and breaks into newly born broken waves which will soon evolve into spirals waves since they have broken ends. This leads to spatiotemporal chaos or irregularity within the excitable medium as shown in Fig.4 where the time variations of the excitatory and the recovery

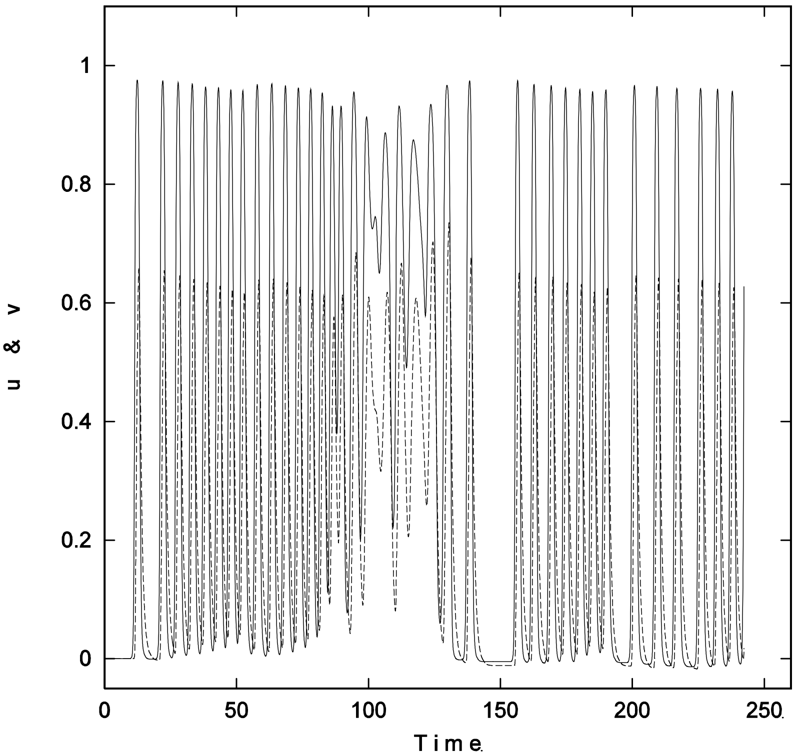

Fig. 4 Time variation of the excitatory and the recovery variable recorded in the medium at the location (10,10). The dotted one is the variation of v.

variables are recorded at point (10,10) in the excitable medium of size L = 100. This phenomenon can be attributed to an unstable focus by considering a traveling wave solution of (1), $u(z)=u(x+ct)$ where c is the wave speed. Substituting this solution into (1) reduces the reaction-diffusion equations to the following ODE system:

$$dw/dt = cw - (1/\pounds)u(1-u)[u-((b+v)/a)]$$
$$du/dt = w$$
$$dv/dt = (1/c)(u^3 - v)\,, \qquad (2)$$

Using parameter values a=0.75 and b=0.06, $\pounds^{-1} = 13.5$, the numerical solution shown in Fig.5 approaches the resting state in an oscillatory manner. For those values of the parameter, the system can have complex eigenvalues implying that the fixed point (0,0,0) is an unstable focus for $c^2 - 4.32 < 0$. This condition is satisfied for some range of the spiral period as it can be seen in the dispersion

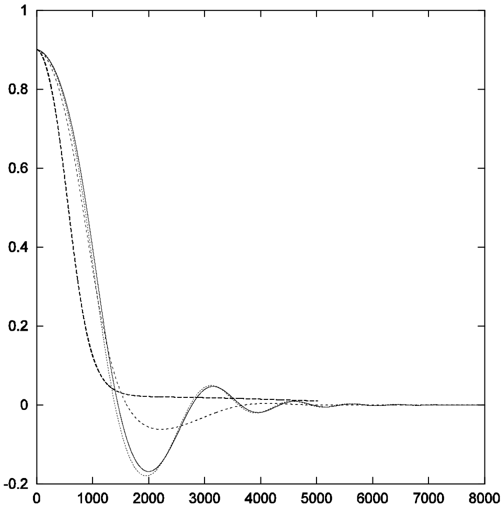

Fig. 5. Recovering solitary traveling solution u(z)=u(x+ct) of Eqs.(2), illustrating damped oscillatory ($£^{-1} = 13.5$, 14.5) and monotonic recovery ($£^{-1} = 20.0$, 50.0).

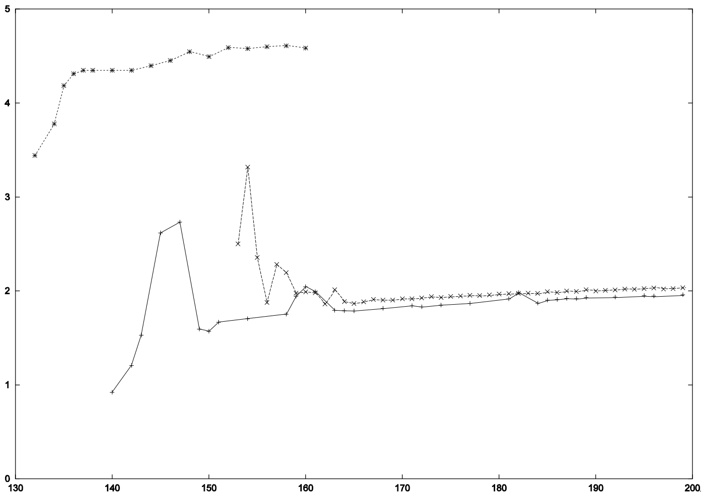

Fig. 6. Dispersion curve: Instantaneous wave speed as a function of period for $£^{-1} = 13.5$, 14.5, 50.0 respectively from bottom to top, computed by repetitively stimulating at one of the ends of an open line.

curve in Fig. 6. That justifies the oscillatory behavior of the numerical solution of (2). For other values of the parameter £ , the solution approaches the resting state (0,0,0) in a non-oscillatory manner if $£^{-1} = 50.0$ for which the spiral rotates rigidly around a circular core. If $£^{-1} = 20.0$, the recovery is also monotonic, but that does not necessarily imply rigid rotation. Actually, the spiral tip meanders following an epicycle-like orbit as shown in Fig. 3(b). On the other hand, if $£^{-1} = 13.5$ or $£^{-1} = 14.5$, the solution returns to the resting state in an oscillatory manner as seen in Fig. 5; the system undergoes a succession of super and subnormal periods until complete recovery is achieved. However, for $£^{-1} = 14.5$, unlike the case for $£^{-1} = 13.5$ (shown in Fig. 3(a) and Fig. 2) and despite the oscillatory type of recovery, core expansion does not occur and the tip does not follow an outward spiraling trajectory. It traces loops like those of an epicycle as the spiral wave rotates and drifts away. Actually, we found that the tip moves along a spiraling path in the range 13.0<$£^{-1}$<13.8. For $£^{-1}$>13.9, it meanders but not along a spiraling trajectory. Thus, we note the important conclusion that oscillatory recovery does not necessarily lead to core expansion and spiraling tip.

This oscillatory recovery can be further investigated by writing the solution of the reaction-diffusion system as a superposition of two solitary waves with a small perturbation term R which vanishes in the limit of infinite spacing between the two waves:

$$u(z) = u(z - z_1) + u(z - z_2) + R \qquad (3),$$

where $z_1 = x_1 - ct$ and $z_2 = x_2 - ct$, x_1 and x_2 denote the waves positions. For large $|z_1|$ and $|z_2|$, the tail of the wave determines the manner in which the medium recovers to the resting state is: When the recovery is damped oscillatory, the tail of the wave $u(z) \propto e^{¦ z} \cos(¬z + ·)$; when it is monotonic, $u(z) \propto e^{¦ z}$. In both cases, the leading edge of the wave is assumed to be of the form $u(z) \propto e^{-« z}$. Equations for x_1 and x_2 are derived using the solvability conditions which remove singularities from R [21]:

$$\frac{¢x_1}{¢t} = c + a_R e^{-«(x_1 - x_2)} \qquad (4)$$

$$\frac{¢x_2}{¢t} = c + a_L e^{-¦(x_1 - x_2)} \cos(¬(x_1 - x_2) + ·) \qquad (5),$$

where c is the propagation speed of a solitary wave. The second term on the right hand side of (4) represents the effect of the second wave on the propagation of the first one. It is usually negligible in excitable media. The second term on the right hand side of (5) represents the effect exerted on the second wave by the refractory wake of the first one. Using (4) and (5), the spacing between the two waves $ª = x_1 - x_2$ obeys the equation:

$$\frac{d\text{a}}{dt} = a_L e^{-¦\ \text{a}} \cos(¬\text{a} + \cdot\ ) \qquad (6).$$

If $¬ \neq 0$, the excitable medium recovers in an oscillatory way. Then, according to (6) an infinite number of steady state solutions exist. This means that the distance between the wave-fronts takes one of possible fixed values.

Those oscillations in the way of recovery to the resting state would imply oscillations in the dispersion. This is shown here by considering the times when wave-fronts pass through a given location x. The solution of (5) is then approximated by widely spaced impulses:

$$u(x,t_i) = \sum_k u_k(t_i(x)) + R, \qquad (7)$$

where $t_i(x)$ is the instant at which the ith impulse is at x, and R is a small perturbation term which vanishes in the limit of infinite spacing between the waves. Using (7) in (1), we get

$$dt_i / dx = (1/c_0) + a'e^{¦(t_i - t_{i-1})} \cos[v(t_i - t_{i-1}) + \cdot\ ] + b'e^{-«(t_{i+1} - t_i)}$$

(8)

where η is the rate at which the wave-fronts tail off. The second term on the right hand side of (8) represents the effect exerted on the ith impulse by the refractory wake of the preceding impulse. The last term represents the effect of the succeeding impulse and is negligible in excitable media. The coefficients a' and b' require the evaluation of certain integrals which are not shown here. Let $t_i(x) = (x/c) + (i-1)T$, where T is the period of a constant speed wave-train. Then we get to leading order,

$$c = c_0 - c_0^2 a'e^{-¦\ T} \cos[¬T + \cdot\ ] \qquad (9)$$

For ν ≠ 0 in (9), damped oscillations occur in the dispersion curve. For monotonic recovery (ν = 0, a>0), the wave speed is a monotonic increasing function of wave spacing. If ν = 0 and a<0, the recovery is said to be non-monotonic and could exhibit one supernormal period [10, 19]. In Fig. 6, there are damped oscillations in dispersion curve of the system (1) for $£^{-1} = 13.5$ and $£^{-1} = 14.5$. The first supernormal period during which the excitability is higher than that of the rest state is very pronounced. However, for $£^{-1} = 50.0$, as expected, the monotonically recovering system is characterized by a dispersion curve with monotonic increase in the propagation velocity until the limit set by the solitary wave velocity is reached.

3. Conclusions

This oscillatory behavior and the occurrence of supernormal periods in the dispersion curve were observed in wave train solutions of the one-dimensional FitzHugh-Nagumo model [23]. But, no core expansion has ever been reported before. Also, the observation of expanding cores and spiraling tips here answers the query of Meron [6, 9, 10] and Winfree [24, 25] about the possible observation of core expansion and oscillations in the dispersion curve. Using FitzHugh-Nagumo kinetics with parameters chosen such that the equilibrium point is nearly a center, Winfree showed that the medium can then support two stable rotors of different periods. The dispersion curve exhibited a damped oscillatory behavior. However, core expansion was not observed, and meandering along a spiraling path was not obtained.

This occurrence of 'supernormal' periods of excitability during which the threshold of excitation is diminished was reported in electrophysiological measurements in stimulated cardiac muscle [26]. The current that was needed to re-excite the Purkinje fibers was reduced. We could attribute it to the faster recovery of the threshold potential compared to the slower recovery of the action potential that we have seen here. We have also verified that than a smaller additional depolarization is needed to reach the threshold potential and it was brought about by a weaker depolarizing current.

Our results would imply that core expansion could be one possible route to spiral breakup.

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