Chaos Synchronization and Chaos Control Based on Kannan Mappings

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Abstract: In this paper, a new method for constructing chaotically synchronizing systems is proposed. Furthermore, a new control method for stabilizing a periodic orbit embedded in a chaotic attractor is proposed. The validity of these methods is shown by a property of Kannan mappings. It is shown that in some cases in which method of contraction mappings, proposed by Ushio (T. Ushio. Chaotic Synchronization and Controlling Chaos Based on Contraction Mappings, *Physics Letters A*, vol. 198, 14-22, 1995.), cannot be applied to synchronize or control of chaotic systems, the method may be applied. Ultimately, a numerical example is given in order to present the results established.
Keywords: chaos synchronization, chaos control, Kannan mappings.

1 Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively studied over the past decades. Dynamic chaos has aroused considerable interest in many areas of science and technology due to its powerful applications in chemical reactions, power converters, biological systems, information processing, secure communication, neural networks etc. In the study of chaotic systems, chaos synchronization and chaos control play a very important role and have great significance in the application of chaos.

Chaos synchronization seems to be difficult to observe in physical systems because chaotic behavior is very sensitive to both the initial conditions and noise. However, Pecora and Carroll [1] have successfully proposed a method to synchronize two identical chaotic systems with different initial conditions. Since then, a variety of approaches have been proposed for the synchronization of chaotic systems which include contraction mappings [2], variable structure control [3,4], parameters adaptive control [5,6], observer based control [7,8], nonlinear control [9-11], nonlinear replacement control [12], variable strength linear coupling control [13], active control [14,15] and so on.

On the other hand, chaos control is a very attractive subject in the study of chaotic systems. Since the method for controlling of chaos was first proposed by Ott et al [16], many chaos control methods have been developed extensively over the past decades such as contraction mappings [2], chaotic targeting

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method [17,18], delayed feedback control [19] etc. Yu et al [20] used the contraction mapping method, proposed by Ushio [2], to stabilize chaotic discrete neural networks.

Neural networks have been widely used as models of real neural structures from small networks of neurons to large scale neurosystem. In recent years, investigation of chaotic dynamics in neural networks becomes an active field in the study of neural networks dynamics. Numerous chaotic neural network models have been proposed for investigation [20-22]. Among the spectrum of applications of chaos control, neural system is a particularly interesting research object of complex structures that it can be applied [23,24].

In this paper, a new method for synthesis of chaotically synchronizing systems based on Kannan mappings is proposed. Also, a new method based on these mappings to stabilize chaotic discrete systems is proposed. These methods are applied to synchronize and control chaotic discrete neural networks. A similar advantage of the methods proposed in this paper and the methods proposed by Ushio [2] is that the linearization of the system near the stabilized orbit is not required. However, in some cases in which the proposed methods of Ushio [2] are not applicable to synchronize or control chaotic systems, the methods may be applied.

This paper is organized as follows. In section 2, problem of chaos synchronization is studied. In section 3, problem of controlling chaos is discussed. Eventually, a numerical example is given in order to present the result investigated.

2 Chaos Synchronization

First, the following theorem which Kannan proved in 1969 is introduced.

Theorem [25] Let (X,d) be a complete metric space. Let T be a Kannan mapping on X, that is, there exists $\alpha \in [0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le \alpha(d(Tx,x)+d(Ty,y))$$

for all $x,y \in X$. Then, there exists a unique fixed point $x_0 \in X$ of T.

We now consider chaotic discrete-time systems described by

$$x(k+1) = f(x(k)), \quad (1)$$

where $x(k) \in \Re^n$ is the state of the system at time k, and f is a mapping from $\Re^n$ to itself. We assume that f is rewritten as follows

$$f := g + h, \quad (2)$$

where both g and h are mappings from $\Re^n$ to itself and g is a Kannan mapping on a closed set $\Omega \in \Re^n$. It is assumed that a chaotic attractor A of Eq. (1) is in Ω. Many methods for constructing synchronized chaotic systems are based upon the decomposition of states of chaotic systems, and it is proved by using conditional Lyapunov exponents whether the constructed systems are synchronized. Ushio proposes a method based on the partition of the nonlinear mapping, and synchronization of the constructed systems is guaranteed by a property of contraction mappings.

This paper proposes another method based on partitioning of the nonlinear mapping, and synchronization of the constructed systems is guaranteed by a property of Kannan mappings. In the following subsections, we study synthesis methods for in-phase and anti-phase synchronization of chaotic systems.

2.1 In-phase synchronization

That the difference of the states of two systems converges to zero is called in-phase synchronization or synchronization. We construct a system described by

$$w(k+1) = g(w(k)) + h(x(k)), \quad (3)$$

where $w(k) \in \Re^n$ is the state of the system, and $x(k) \in \Re^n$ is the state of Eq. (1). Suppose that initial state $x(0)$ of Eq. (1) is in the basin of the attractor A, and both states $x(k)$ and $w(k)$ of Eq. (1) and (3) are in Ω for each $k \in \mathrm{N}$, where N denotes the set of all natural numbers. We assume that there exist a closed set $\Omega \in \Re^n$ and a nonnegative constant $0 \le \alpha < \frac{1}{2}$ such that for any $x, y \in \Omega$ the mapping g satisfies

$$\|g(x) - g(y)\| \le \alpha(\|x - g(x)\| + \|y - g(y)\|).$$

We show that Eq. (1) and (3) are in-phase synchronized, so

$$\|x(k+1) - w(k+1)\| = \|g(x(k)) - g(w(k))\|$$
$$\le \alpha(\|x(k) - g(x(k))\| + \|w(k) - g(w(k))\|).$$

According to Theorem, we obtain

$$\lim_{k\to\infty}\|x(k) - w(k)\| = 0.$$

Thus, in-phase chaotic synchronization of Eqs. (1) and (3) is achieved. Note that $w(0)$ is not necessarily in the basin of A.

Let us consider the following fully connected network composed of m-neurons, as given in [20]:

$$x_{k+1}^i = \varphi_\mu(\sum_{j=1}^m W_{ij} x_k^j), \; i = 1,2,\ldots,m$$

where $\varphi_\mu(z) = (1 + e^{-\mu z})^{-1}$ is assumed to be the sigmoid function. Let $m = 2$, i.e., consider the case where we have a 2D fully connected neural network defined as

$$x_{k+1} = \varphi_\mu (W_{11} x_k + W_{12} y_k),\ (4-a)$$
$$y_{k+1} = \varphi_\mu (W_{21} x_k + W_{22} y_k),\ (4-b)$$

Altering the matrix $W = (w_{ij})$ of connecting, this map can generate various complex dynamical patterns, including deterministic chaos [23]. We start our study with a 2D neural network with matrix

$$W = \begin{pmatrix} -a & a \\ -b & b \end{pmatrix}.$$

This simplified neural network is dynamically equivalent to a one-parameter family of s-unimodal maps; it is well known that this map will generate chaotic via the Feigenbaum scenario.

We partition the neural network as follows

$$h(x_k, y_k) = \begin{pmatrix} \varphi_\mu (w_{11} x_k + w_{12} y_k) - \sqrt{|x_k|} & 0 \\ 0 & \varphi_\mu (w_{21} x_k + w_{22} y_k) - \sqrt{|y_k|} \end{pmatrix},$$

$$g(x_k, y_k) = \begin{pmatrix} \sqrt{|x_k|} & 0 \\ 0 & \sqrt{|y_k|} \end{pmatrix}.$$

The mapping g satisfies Kannan mapping for any $x, y \in \Re$. Then, we have the following new system

$$w_1(k+1) = \varphi_\mu (w_{11} x_k + w_{12} y_k) - \sqrt{|x_k|} + \sqrt{|w_1(k)|},\ (5-a)$$
$$w_2(k+1) = \varphi_\mu (w_{21} x_k + w_{22} y_k) - \sqrt{|y_k|} + \sqrt{|w_2(k)|}.\ (5-b)$$

So in-phase synchronization of System (4) and System (5) is achieved.

Remark 1 Because $\sqrt{|x|}$, $x \in \Re$ is not contraction mapping, the results given in [2] are not applicable to show the synchronization of System (4) and System (5).

2.2 Anti-phase synchronization

That the states of synchronized systems have the same absolute values but opposite signs is called anti-phase synchronization. We can say that anti-phase synchronization holds if

$$\lim_{k\to\infty}\|x_1(k)+x_2(k)\|=0,$$

where $x_i, i=1,2$, is the state of the system. Suppose that the state $x(k)$ is both in the basin of the chaotic attractor A and in Ω, and $w(k)$ is in Ω.
Then,

$$\begin{aligned}\|x(k+1)+w(k+1)\| &= \|g(x(k))+g(w(k))\| \\ &\le \alpha(\|x(k)+g(x(k))\|+\|w(k)+g(w(k))\|)\end{aligned}$$

According to Theorem, we obtain

$$\lim_{k\to\infty}\|x(k)+w(k)\|=0.$$

Thus, anti-phase chaotic synchronization of $x(k)$ and $w(k)$ is achieved.

3 Chaos Control

Consider the following chaotic discrete-time systems with an external input

$$Z_{k+1}=f(Z_k)+Bu_k, \quad (6)$$

where $Z_k\in\Re^n$ and $u_k\in\Re^l$ are the state and input of the system, and B is an $n\times l$ constant matrix. Eq. (6) without input has a chaotic attractor A. Let $Z^*=f(Z^*)$ be a periodic orbit embedded in A. We consider the following input

$$u_k=\begin{cases}D(z_k)-D(z^*) & if\ \|z_k-z^*\|<\varepsilon \\ 0 & otherwise\end{cases}, \quad (7)$$

where D is a mapping from $\Re^n$ to $\Re^l$, and ε is a sufficiently small positive constant. Assume that the mapping $f+BD$ is a Kannan mapping on a closed set $\Omega\in\Re^n$, and the chaotic attractor A is within Ω. Suppose that the initial state z_0 of Eq.(6) is within Ω; then, the following behavior z_k controlled by Eq.(7) is expected

$$\begin{aligned}\|z_{k+1}-z^*\| &= \|(f+BD)z_k-(f+BD)z^*\| \\ &\le \alpha(\|z_k-(f+BD)z_k\|+\|z^*-(f+BD)z^*\|).\end{aligned}$$

Since $0\le\alpha<\frac{1}{2}$, according to Theorem, we get $\lim_{k\to\infty}\|z_k-z^*\|=0$, and the periodic orbit z^* can be stabilized in Ω.

As in [20], we consider the neural network defined as follows:

$$x_{k+1} = \varphi_\mu(w_{11}x_k + w_{12}y_k) + u_{1k}\,, \quad (8-a)$$

$$y_{k+1} = \varphi_\mu(w_{21}x_k + w_{22}y_k) + u_{2k}\,, \quad (8-b)$$

where $u_{1k}, u_{2k} \in \Re$ are control inputs. Then, we have

$$z_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}, f(x_k, y_k) = \begin{pmatrix} \varphi_\mu(w_{11}x_k + w_{12}y_k) \\ \varphi_\mu(w_{21}x_k + w_{22}y_k) \end{pmatrix}$$

and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now, let us consider the following mapping

$$D(x_k, y_k) = \begin{pmatrix} -\varphi_\mu(w_{11}x_k + w_{12}y_k) + \sqrt{|x_k|} \\ -\varphi_\mu(w_{21}x_k + w_{22}y_k) + \sqrt{|y_k|} \end{pmatrix}.$$

Then, the mapping $f + BD$ is a Kannan mapping. Thus, the following control input can stabilize any periodic orbit embedded in a chaotic attractor of (6)

$$u_k = \begin{cases} \begin{pmatrix} u_{1k} \\ u_{2k} \end{pmatrix} = \begin{pmatrix} -\varphi_{1\mu}(x_k, y_k) + \sqrt{|x_k|} + \varphi_{1\mu}(x^*, y^*) - \sqrt{|x^*|} \\ -\varphi_{2\mu}(x_k, y_k) + \sqrt{|y_k|} + \varphi_{2\mu}(x^*, y^*) - \sqrt{|y^*|} \end{pmatrix} & \text{if } \|z_k - z^*\| < \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

where $z^* = (x^*, y^*)$ denotes a stabilized periodic state with period 1. To obtain the necessary information of an approximate location of the desired periodic orbit, the strategy described in Ref. [26] is utilized. We collect a long data string of observed $z_1, z_2 = f(z_1)$ and so on. If two successive z_3 are closed to each other, say z_{100} and z_{101}, then there will typically be a period-1 orbit z^* nearby. Having observed a first such close return pair, we then search the succeeding data for other close return pairs (z_k, z_{k+1}) restricted to the small region of the original close return. Because orbits on a strange attractor are ergodic, we will get many such pairs if the data string is long enough. When the first close return pair is detected, the first point of the pair is taken as a reference point. There are a number of close return pairs detected, which are close to reference point, where $z_{j,1}$ and $z_{j,2}$ are respectively used to denote the first point and its successive point of the jth collected return pair, $j = 1,2,\ldots,M$, where M is the maximum number of collected return pairs. The mean value

$$z^* = \frac{1}{2M}\sum_{j=1}^{M}(z_{j,1} + z_{j,2}) \text{ , (9)}$$

is regarded as an approximate fixed point z^*. This fixed point can be used to define a neighborhood $|z_i - z^*| \le \varepsilon$ in which control input is activated.

Remark 2 In comparison with the results given in [20], it can be seen that using controller u_k, proposed in this section, the results of [20] cannot show the control of the chaotic discrete neural network.

3 Numerical Example

Consider the following chaotic neural network

$$x_{k+1} = \varphi_\mu(-5x_k + 5y_k) + u_{1k} \text{ , } \quad (10-a)$$

$$y_{k+1} = \varphi_\mu(-25x_k + 25y_k) + u_{2k} \text{ , } \quad (10-b)$$

where $\varphi_\mu(z) = (1 + e^{-\mu z})^{-1}$ is assumed to be the sigmoid function. The system has chaotic behavior for $\mu = 5.5$, and the approximate period-3 orbit is estimated at $(0.999496, 1.00000)^T, (0.593963, 0.870103)^T$ and $(0.503459, 0.517291)^T$, when the condition $|z_i - z_{i+2}| \le 0.005$ is satisfied [20].

We first show the simulation results of chaotically synchronizing System (10) and System (5) without control input. So System (5) becomes as follows

$$w_1(k+1) = \varphi_\mu(-5x(k) + 5y(k)) - \sqrt{|x(k)|} + \sqrt{|w_1(k)|}, \quad (11-a)$$

$$w_2(k+1) = \varphi_\mu(-25x(k) + 25y(k)) - \sqrt{|y(k)|} + \sqrt{|w_2(k)|}, \ (11-b)$$

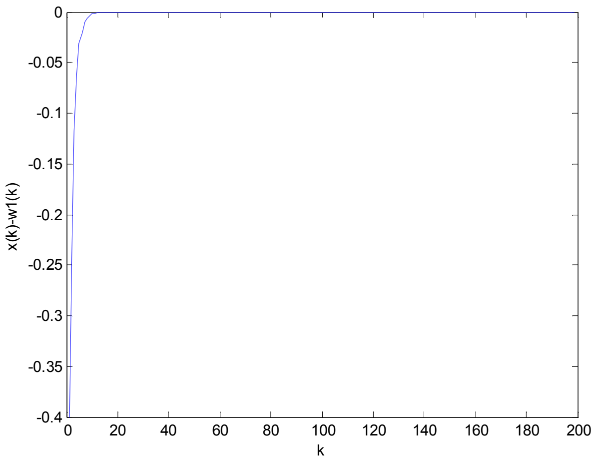

Fig. 1. The error $x(k) - w_1(k)$

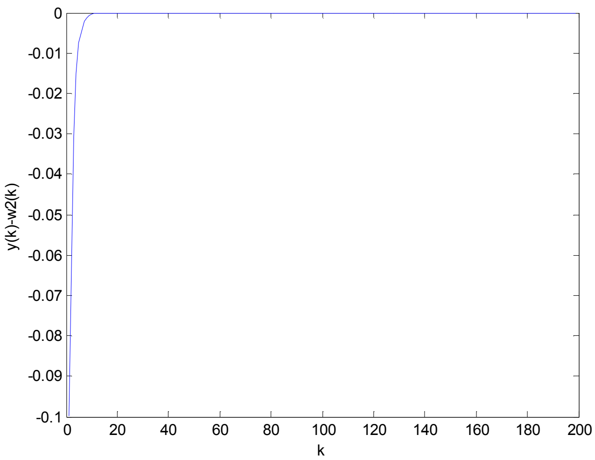

Fig. 2. The error $y(k) - w_2(k)$

The system is simulated with initial conditions $x(0) = 0.5, y(0) = 0.6, w_1(0) = 0.9, w_2(0) = 0.7$, and the differences are showed in Figs. (1) and (2). These figures show that system (10) is synchronized with system (11).

Now, we show the simulation results of chaos control of System (10) using controller u_k proposed in previous section.

Behaviors of the state variables x and y and the input controls u_1 and u_2 are shown in Figs. 3-6, when a periodic orbit with perio d=3 is stabilized with $\varepsilon = 0.002$.

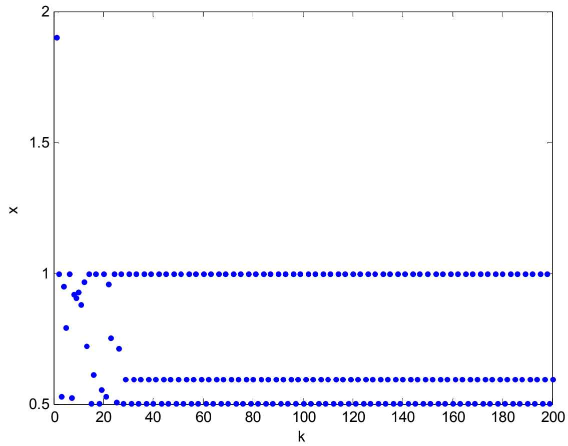

Fig. 3. Behavior of *x*.

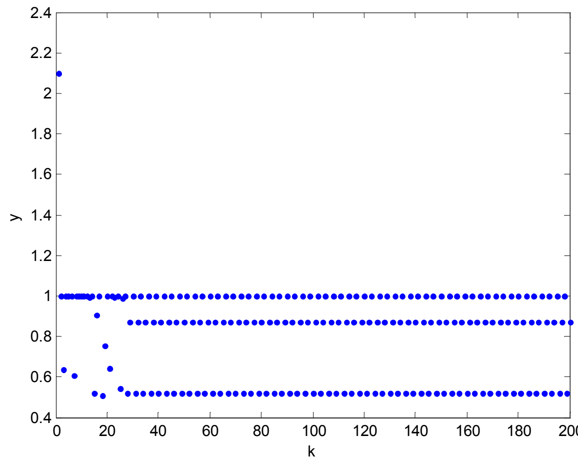

Fig. 4. Behavior of *y*.

Figs.3 and 4 show behaviors of the state variables x and y, respectively, with initial condition $(1.9 \quad 2.1)^T$.

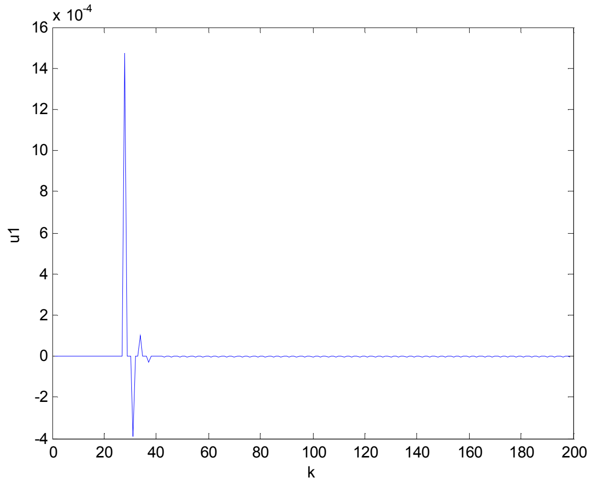

Fig. 5. Behavior of input control u_1.

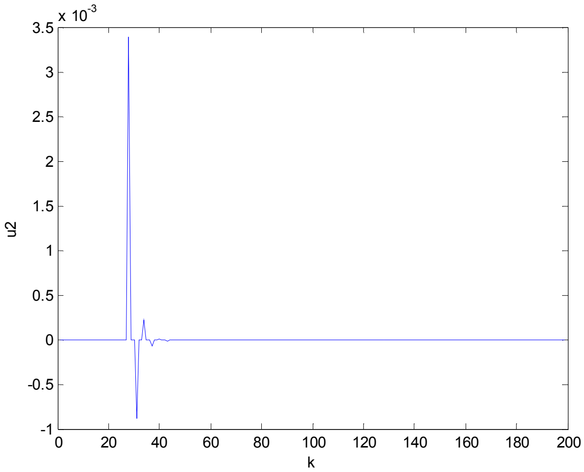

Fig. 6. Behaviorf of input control u_2.

Figs.5 and 6 show behaviors of the input controls u_1 and u_2, respectively. These figures show that System (10) is stabilized by the controller proposed in this paper.

5 Conclusions

In this paper, a new method based on Kannan mappings for chaotic synchronization is proposed. Furthermore, a new method based on the mappings is presented to stabilize chaotic discrete systems. These methods are applied to synchronize and control of chaotic discrete neural networks. Finally, a numerical example is given to validate the methods presented.

References

1. L. M. Pecora and L. Carroll. Synchronization in Chaotic Systems, *Physical Review Letters*, vol. 64, 821-824, 1990.
2. T. Ushio. Chaotic Synchronization and Controlling Chaos Based on Contraction Mappings, *Physics Letters A*, vol. 198, 14-22, 1995.
3. L. Liao and N. S. Huang. Control and Synchronization of Discrete Time Chaotic Systems via Variable Structure Control Technique, *Physics Letters A*, vol. 234, 262-268, 1997.
4. C. Yang, Z. Ge, C. Chang, and S. Li. Chaos Synchronization and Chaos Control of Quantum-CNN Chaotic System by Variable Structure Control and Impulse Control, *Nonlinear Analysis, Real World Applications*, vol. 11, 1977-1985, 2010.
5. D. Vassiliadis. Parametric Adaptive Control and Parameter Identification of Low Dimensional Chaotic Systems, *Physica D: Nonlinear Phenomena*, vol. 71, 319-341, 1994.
6. S. Chen and J. Lu. Parameters Identification and Synchronization of Chaotic Systems Based upon Adaptive Control, *Physics Letters A*, vol. 299, 353-358, 2002.
7. E. Solak, O. Morgul, and U. Ersoy. Observer-Based Control of A Class of Chaotic Systems, *Physics Letters A*, vol. 279, 47-55, 2001.
8. J. A. Ramirez, H. Puebla, and I. Cervantes. Convergence Rate of Observer-Based Approach for Chaotic Synchronization, *Physics Letters A*, vol. 289, 193-198, 2001.
9. L. Huang, R. Feng, and W. Wang. Synchronization of Chaotic Systems via Nonlinear Control, *Physics Letters A*, vol. 320, 271-275, 2004.
10. J. H. Park. Chaos Synchronization of A Chaotic System via Nonlinear Control, *Chaos, Solitons, and Fractals*, vol. 25, 579-584, 2005.
11. H. K. Chen. Global Chaos Synchronization of New Chaotic Systems via Nonlinear Control, *Chaos, Solitons, and Fractals*, vol. 23, 1245-1251, 2005.
12. R. Tang and J. Xue. Chaotic Synchronization by Replacing Nonlinear Terms with Signals, *Chaos, Solitons, and Fractals*, vol. 28, 228-235, 2006.
13. Z. Ge and P. Tsen. Two Theorems of Generalized Unsynchronization for Coupled Chaotic Systems, *Nonlinear Analysis, Theory, Method, and Applications*, vol. 69, 4230-4240, 2008.
14. H. N. Agiza and M. T. Yassen. Synchronization of Rossler and Chen Chaotic Dynamical Systems Using Active Control, *Physics Letters A*, vol. 278, 191-197, 2001.
15. M. T. Yassen. Chaos Synchronization Between Two Different Chaotic Systems Using Active Control, *Chaos, Solitons, and Fractals*, vol. 23, 131-140, 2005.

16. E. Ott, C. Grebogi, and J. A. Yorke. Controlling Chaos, *Physical Review Letters*, vol. 64, 1190-1199, 1990.
17. M. Kopel. Improving The Performance of An Economic System: Controlling Chaos, *Evelutionary Economics*, vol. 7, 269—289, 1997.
18. L. Kass. Stabilizing Chaos in A Dynamical Macroeconomic Model, *Economic Behavior and Organization*, vol. 33, 313-332, 1998.
19. J. A. Holyst and K. Urbanowicz. Chaos Control in Economical Model by Time Delayed Feedback Method, *Physica A*, vol. 287, 587-598, 2000.
20. H. Yu, Y. Liu, and J. Peng. Control of Chaotic Neural Networks Based on Contraction Mappings, *Chaos, Solitons, and Fractals*, vol. 22, 787-792, 2004.
21. K. Aihara, T. Takabe, and M. Toyoda. Chaotic Neural Network, *Physics Letters A*, vol. 144, 333-340, 1990.
22. S. Nara, P. Davis, and H. Totsuji. Memory Search Using Complex Dynamics in A Recurrent Neural Network Model, *Neural Networks*, vol. 6, 963-973, 1993.
23. R. Sole, L. Menendez, and D. Prida. Controlling Chaos in Discreteneural Networks, *Physics Letters A*, vol. 199, 65-69, 1995.
24. H. Sabbagh. Control of Chaotic Solutions of The Hindmarsh-Rose Equations, *Chaos, Soliton, and Fractals*, vol. 112, 1213-1218, 2000.
25. R. Kannan. Some Results on Fixed Points-II, *American Mathematical Monthly*, vol. 76, 405-408, 1969.
26. T. Shinbrot, C. Grebogi, E. Ott, and J. York. Using Small Perturbations to Control Chaos, *Nature*, vol. 363, 411-417, 1993.