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Shadow Prices and Lyapunov Exponents

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Abstract: A relation between the optimal solution of the optimization problem and the stability and bifurcation properties of the corresponding dynamical system is suggested in this work. There exists a relation between the optimal solution of an optimization problem and an equilibrium point of a dynamical system. In this sense stability properties, Lyapunov exponents and bifurcations of the resulting dynamical systems can be studied.
Keywords: Dynamical systems, Optimization, Lyapunov exponents.

1. Introduction:

Shadow price is the unit change in the objective function of the optimal solution of an optimization problem. The shadow price is equivalent to the Lagrange multiplier at the optimal solution in the nonlinear scenario. It is also referred to as the dual variable considering the Lagrangian is the dual problem of the original optimization problem. The gradient of the objective function is a linear combination of the constraint function gradients with the weights equal to the Lagrange multipliers. Investigations on various linear optimization problems can be formulated as dynamical systems [4]. Stability analysis, Lyapunov exponents and bifurcation patterns of the resulting dynamical systems can be studied in a localized manner [2]. There is a relation between the global optimum value of the optimization problem to the local stability analysis of the corresponding dynamical system. The bifurcation properties and Lyapunov exponents of the corresponding dynamical system can be studied. The aim is to compare these invariant parameters of the dynamical systems to the shadow prices of the optimization problem. The motivation for this is the fact that to calculate a Lyapunov exponent, each dynamical variable is given a small variation and the corresponding hypercube is allowed to evolve in time [1]. Let us start by defining an optimization problem as

$$\max\{f(x,y): ax + by = c\}$$

Then the Lagrangian function is given by (in the two variable case)

$$L(x,y,\lambda) = f(x,y) + \lambda(c - ax - by)$$

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(with obvious generalization to higher dimensions) and by solving this function for its saddle point we obtain the shadow prices and the maximal utility, x^*, y^*, λ^*, given by the following formula:

$$\lambda^* = \frac{\frac{\partial f(x^*,y^*)}{\partial x}}{a} = \frac{\frac{\partial f(x^*,y^*)}{\partial y}}{b}$$

On the other hand, shadow prices are found by observing the change in the optimal solution under a similar variation on the constraint of the direct problem by relaxing the constraint or alternatively, varying the corresponding parameter of the objective function in the dual problem. The definitions for the Lyapunov exponents and shadow prices are thus related to a change due to a variation. The former is a familiar element of the theory of dynamical systems. The route to chaos leads to Lyapunov exponents and this work introduces a new point of view for shadow prices as chaos search in dynamical systems [3]. Under the assumption that f be differentiable and $y_0 \neq 0$ the variational equation is:

$$y_{t+1} = \frac{df(x_t)}{dx} y_t$$

Then the Lyapunov exponent is defined to be

$$\lambda(x_0, y_0) = \lim_{n \to \infty} \frac{\ln \left| \frac{y_n}{y_0} \right|}{n}$$

A negative Lyapunov exponent indicates a stable equilibrium point and a positive Lyapunov exponent indicates chaos. So Lyapunov exponents are studied numerically to see if the given system shows chaos for certain parameter values. It has been proven that discrete-time dynamical systems are used in optimization algorithms. We also know that a discrete-time dynamical system can be transformed into a continuous dynamical system, i.e. system of differential equations by Euler's method. Both proofs depend on Lyapunov stability theory.

2. Optimization problem and corresponding dynamical system

Theorem 2.1: For the optimization problem

$$\max f(x,y) = x^k + y^k$$

with respect to $g(x,y) = 1 - x - y$

the extremum values are $(x^*, y^*) = (\frac{1}{2}, \frac{1}{2})$ and the general term of each Lagrange multiplier is $\lambda^* = -\frac{k}{2^{k-1}}$.

Proof:

$$\nabla f = \{kx^{k-1}, ky^{k-1}\}$$
$$\nabla g = \{-1,-1\}$$
$$k\begin{bmatrix} x^{k-1} \\ y^{k-1} \end{bmatrix} = \lambda^* \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$
$$\lambda^* = -k\begin{bmatrix} x^{k-1} \\ y^{k-1} \end{bmatrix}$$
$$x^{k-1} = \frac{1}{2^{k-1}}$$
$$\lambda^* = -\frac{k}{2^{k-1}}$$

3. Bifurcation analysis:

The optimization problem discussed in the previous section can be considered as the corresponding dynamical system according to the Euler scheme:

$$\dot{x} = x^k + y^k - x$$
$$\dot{y} = 1 - ax - by - y$$

Investigating the bifurcation analysis of this system around the trivial equilibrium point, two different bifurcation patterns are achieved according to the value of k being odd or even. The first case where k is even (k=2,4,…) and a is chosen as the bifurcation indicates a limit point (LP) and a Bogdanov-Takens (BT) bifurcation point as given in Figure 2.1. When b is varied another case where a subcritical Hopf bifurcation point and a transcritical bifurcation point are observed as given in Figure 2.2. The second case where k is odd (k=1,3,…) and a is chosen as the bifurcation indicates two limit point (LP), a Bogdanov-Takens (BT) and a cusp (CP) bifurcation points as given in Figure 2.3. When b is varied another case where a subcritical Hopf bifurcation point and a transcritical bifurcation point are observed as given in Figure 2.4.

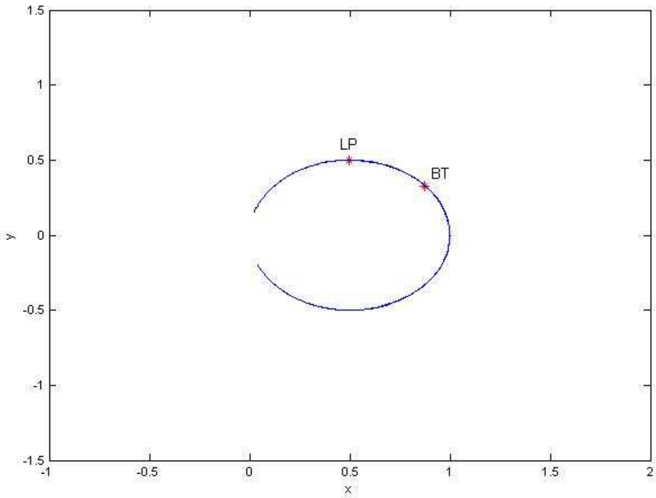

Figure 2.1. For even k (k=2,4,…) and arbitrary a

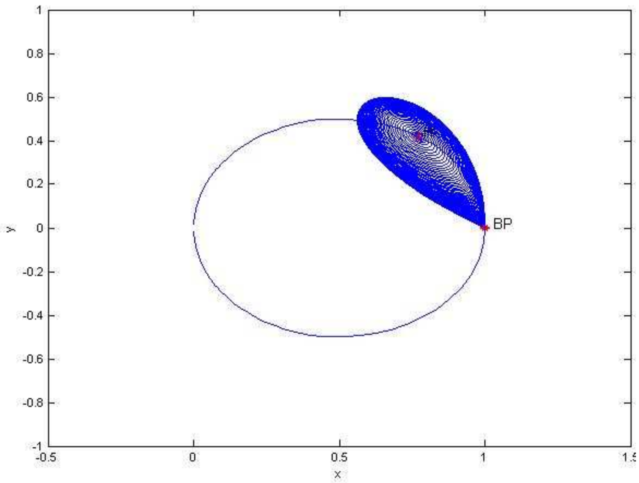

Figure 2.2. For even k(k=2,4,…) and arbitrary b

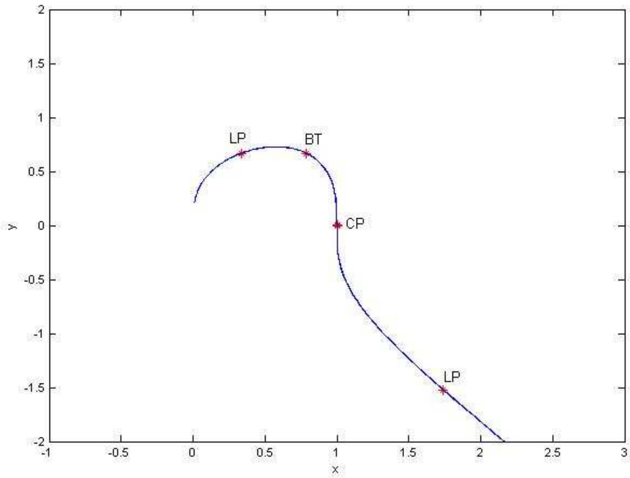

Figure 2.3. For odd *k* (*k*=3,5,…) and arbitrary *a*

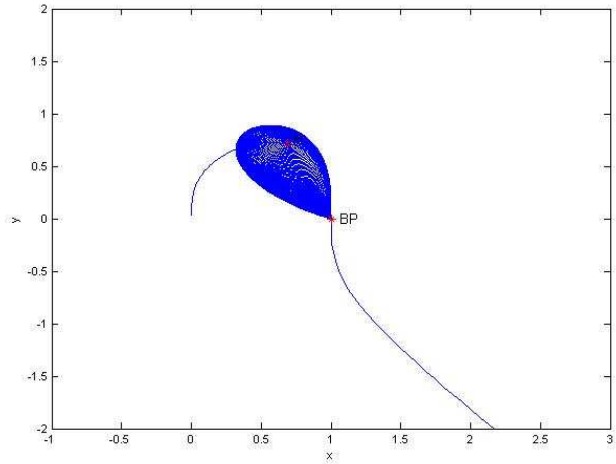

Figure 2.4. For odd *k* (*k*=3,5,…) and arbitrary *b*

4. Conclusion

The parameter b in our model indicates subcritical Hopf bifurcation for both even and odd cases of k. Bogdanov-Takens bifurcation is observed in all of the cases. Cusp bifurcation is observed for odd values of k. The higher nonlinearity for x and y does not affect the bifurcation phenomena. There are two different bifurcation patterns for odd and even values of k. Real values are taken into consideration in order to study real world situations.

References

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