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Chaos in Compound Anti-Symmetric-Case Piecewise-Linear Delay Differential Equations

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Abstract: An existing anti-symmetric-case piecewise-linear delay differential equation (DDE) has exhibited chaos at a delay time $\tau = 3$ using an odd term $f_a = f_1$ for $a = 1$. Three new compound anti-symmetric-case piecewise-linear DDEs are presented. Each DDE exhibits chaos using $\tau < 3$. The first compound DDE is a combination of two odd terms f_1 and f_3 where $a = 1$ and 3, and $1.70 < \tau < 2.10$. The second compound DDE is a combination of two even terms f_2 and f_4 where $a = 2$ and 4, and $1.50 < \tau < 1.90$. Finally, the third compound DDE is a combination of two odd terms f_1 and f_3, and an even term f_2 where $a = 1$, 2, and 3, and $1.05 < \tau < 1.27$. Not only can the higher value of 'a' reduce the value of τ for chaos, but the more combination of terms f_a also can. The reduction in τ enables simple implementation of a LC network in the delay unit.
Keywords: chaos, delay differential equation; reduced-delay

1. Introduction

Since the discovery of the eminent Lorenz chaotic attractor in 1963 [1], studies of chaotic behavior in nonlinear systems have attracted great attention due to a variety of applications in science and technology, e.g. chaos-based secure communications [2], [3], [4]. Time-delay systems can exhibit chaos with a relatively simple model involving a value of the dynamical variable at one or more times in the past [5]. They have an infinite-dimensional state space with a large value of positive Lyapunov exponents and are good candidates for highly secure communications. In general, a first-order time-delay system is described by a delay differential equation (DDE) of the form.

$$\dot{x}(t) = f[x(t), x_\tau] \tag{1}$$

where the overdot denotes a time (t) derivative, $x_\tau = x(t-\tau)$ is the value of x at an earlier time ($t-\tau$), and τ is a delay time, i.e. $\tau \le t$.

One of the earliest and most widely studied DDE is the Mackey-Glass equation [6], as shown in (2), proposed to model the production of white blood cells. The equation exhibits chaos with parameters such as $a = 0.2$, $b = 0.1$, $c = 10$, and $\tau = 23$. Other examples of DDEs exhibiting chaos include Ikeda DDE [7] and sinusoidal DDE [5].

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$$\dot{x} = \frac{ax_\tau}{1 + x_\tau^c} + bx, \tag{2}$$

Recently, chaos in an anti-symmetric-case piecewise-linear DDE has been reported [5], as shown in (3).

$$\dot{x} = |x_\tau + 1| - |x_\tau - 1| - x_\tau \tag{3}$$

for τ = 3. The largest Lyapunov exponent λ = 0.0909. Such a system is especially amenable to implementation with electronic circuits [8]. A delay unit may be implemented using an LC network [9]. As the size of the LC network is proportional to the value of the delay time τ, a reduction of τ in (3) is preferable.

In this paper, three new compound anti-symmetric-case piecewise-linear DDEs are presented. Each DDE exhibits chaos using delay time $\tau < 3$. Such a reduction of the delay time in the DDEs enables simple implementation of the LC network in the delay unit.

2. Compound Anti-Symmetric-Case Piecewise-Linear DDEs

For simplicity, the right hand side of (3) can be modified as a general function f_a as shown in (4)

$$f_a = |x_\tau + a| - |x_\tau - a| - x_\tau \tag{4}$$

where the parameter 'a' is an integer. Equation (3) is therefore represented by an odd term f_1 as a = 1. Three new compound anti-symmetric-case piecewise-linear DDEs are proposed. The first compound DDE is a combination of two odd terms f_1 and f_3 where a = 1 and 3, as shown in (5). The second compound DDE is a combination of two even terms f_2 and f_4 where a = 2 and 4, as shown in (6). Finally, the third compound DDE is a combination of two odd terms f_1 and f_3, and an even term f_2 where a = 1, 2, and 3, as shown in (7).

$$\begin{aligned} \dot{x}_1 &= f_1 + f_3 \\ &= |x_\tau + 1| - |x_\tau - 1| + |x_\tau + 3| - |x_\tau - 3| - 2x_\tau \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{x}_2 &= f_2 + f_4 \\ &= |x_\tau + 2| - |x_\tau - 2| + |x_\tau + 4| - |x_\tau - 4| - 2x_\tau \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{x}_3 &= f_1 + f_2 + f_3 \\ &= |x_\tau + 1| - |x_\tau - 1| + |x_\tau + 2| - |x_\tau - 2| + |x_\tau + 3| - |x_\tau - 3| - 3x_\tau \end{aligned} \tag{7}$$

3. Numerical Results

For the first compound DDE shown in (5), Figures 1, 2 and 3 visualize numerical results of a chaotic waveform, a chaotic attractor, and a bifurcation diagram, respectively, using τ = 2.07. The largest Lyapunov exponent is λ = 0.3112.

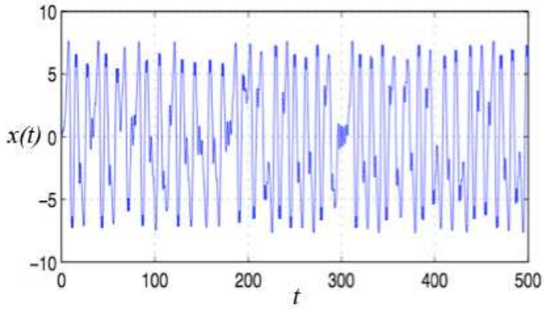

Fig. 1. A chaotic waveform of (5) with τ = 2.07.

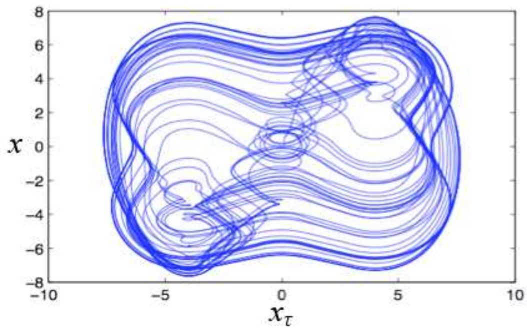

Fig. 2. A chaotic attractor of (5) with τ = 2.07.

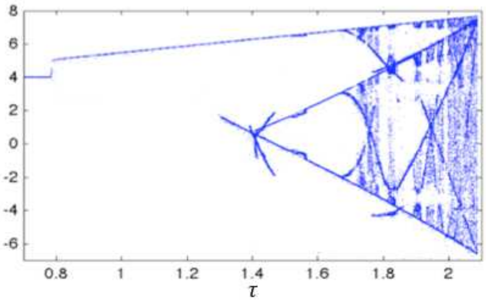

Fig. 3. A bifurcation diagram of (5).

For the second compound DDE shown in (6), Figures 4 and 5 illustrate numerical results of a chaotic attractor and a bifurcation diagram, respectively. (6), using $\tau = 1.75$. The largest Lyapunov exponent is $\lambda = 0.1174$.

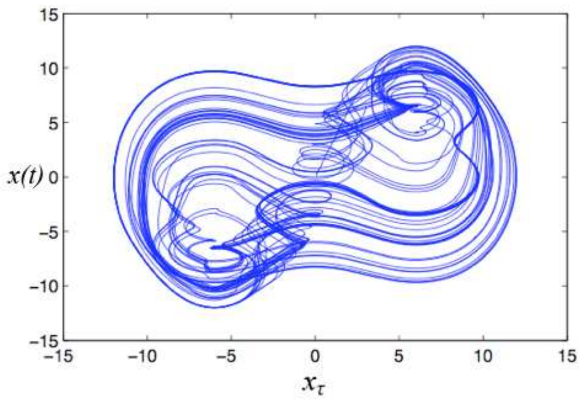

Fig. 4. A chaotic attractor of (6) with $\tau = 1.75$.

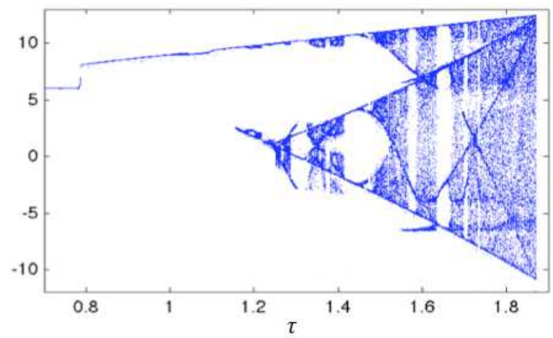

Fig. 5. A bifurcation diagram of (6).

For the third compound DDE shown in (7), Figures 6 and 7 depict numerical results of a chaotic attractor and a bifurcation diagram, respectively, using $\tau = 1.20$. The largest Lyapunov exponent is $\lambda = 0.2823$.

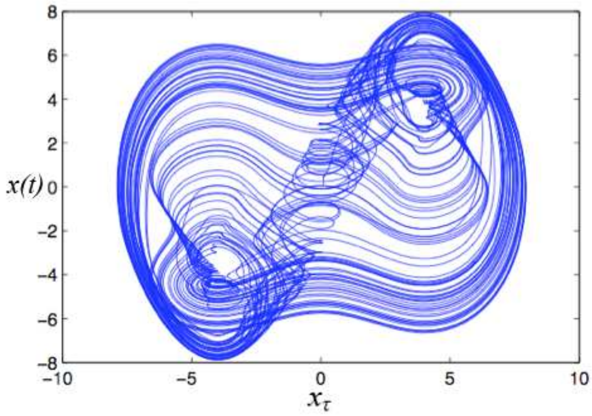

Fig. 6. A chaotic attractor of (7) with τ = 1.20.

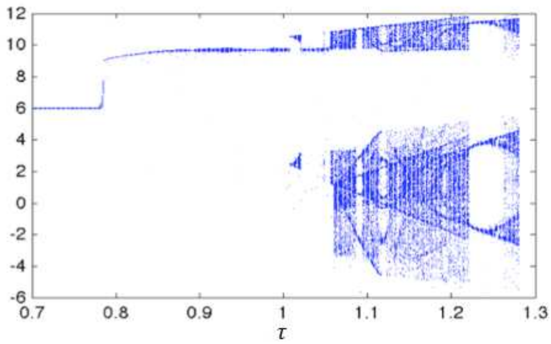

Fig. 7. A bifurcation diagram of (7).

Table 1 summarizes ranges of delay time τ of equations (5), (6), and (7), for which chaos occurs. There are various periodic windows immersed in chaos. It can be notice from Table 1 that not only can the higher value of the parameter 'a' of f_a reduce the value of the time delay τ for chaos, but the more combination of terms f_a also can.

Table 1: Summaries of Ranges of τ For Chaos

Equations	Ranges of τ
$\dot{x}_1 = f_1 + f_3$	$1.70 < \tau < 2.10$
$\dot{x}_2 = f_2 + f_4$	$1.50 < \tau < 1.90$
$\dot{x}_3 = f_1 + f_2 + f_3$	$1.05 < \tau < 1.27$

3. Conclusions

Three new compound anti-symmetric-case piecewise-linear DDEs have been presented. The first combines two odd terms f_1 and f_3 and chaos occurs for $1.70 \angle \tau \angle 2.10$. The second combines two even terms f_2 and f_4 and chaos occurs for $1.50 \angle \tau \angle 1.90$. Finally, the third combines three terms f_1, f_2 and f_3 and chaos occurs for $1.05 \angle \tau \angle 1.27$. Chaos occurs using less delay time τ than that of the existing approach. The reduction in delay time enables the reduction in size of the LC network of the delay unit.

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