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Complex Dynamics and Phase Transitions Caused by Fuzzy Rationality

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Abstract. The notion of dynamical traps is proposed to allow for effect caused by the bounded capacity of human cognition in ordering events or actions according to their preference. As a result, in the vicinity of an optimal behavior a decision-maker has no stimulus to change his current behavior. By way of example, one dimensional system of coupled oscillators with dynamical traps is studied numerically. The model assumes the dynamical traps to form a “low” dimensional region in the corresponding phase space where the system motion is stagnated. It is demonstrated that the dynamical traps and possible noise individually can cause the given system to exhibit complex dynamics and to undergo various phase transitions.
Keywords: Human behavior, Fuzzy rationality, Dynamical traps, Complex dynamics, Phase transitions.

1 Introduction

During the last decades there has been considerable progress in describing social systems based on physical formalism developed in statistical physics and applied mathematics (for a review see articles in Encyclopedia [1]). In particle, the notion of energy and the based on it master equation were employed to simulate opinion dynamics, the dynamics of culture and languages (e.g., [2–4]); the social force model inheriting the basic concepts from Newtonian mechanics was used to simulate traffic flow, pedestrian motion, the motion of bird flocks, fish schools, swarms of social insects (e.g., [2,5–7]). Continuing the list of examples, we note the application of the Lotka-Volterra model and the related reaction-diffusion systems to stock market, income distribution, population dynamics [8]. The replicator equations developed initially in the theory of species evolution were applied to the moral dynamics [9]. The notion of a fixed-point attractor as a stable equilibrium point in the system dynamics that corresponds to some local minimum in a certain potential relief, the collection

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of point type attractors forming a basin, the notion of latent attractors, periodic attractors representing limit cycles, and deterministic chaos are widely met in social psychology [10]. In addition, the concept of synchronization of interacting oscillators was used to model social coordination [11].

In spite of these achievements we have to note that the mathematical theory of social systems is currently at its initial stage of development. Indeed, animate beings and objects of the inanimate world are highly different in their basic features, in particular, such notions as willingness, learning, prediction, motives for action, moral norms, personal and cultural values are just inapplicable to inanimate objects. This enables us to pose a question as to what *individual* physical notions and mathematical formalism should be developed to describe social systems in addition to the available ones inherited from modern physics.

The present paper discusses one of such notions, namely, the fuzzy rationality [12] introduced here to describe the bounded capacity of human cognition in evaluating events, actions, etc. according to their preference. When, for example, two actions are close to each other in quality from the standpoint of a person making a decision their choice may be random because he ought to consider them equivalent. The notion of dynamical traps accounts for this feature. In particular, dealing with a dynamical system its stationary point $\mathbf{r}_{\mathsf{st}}$ being initially stable is replaced by a certain neighborhood $\mathbf{Q}_{\mathsf{tr}}$ called the dynamical trap region such that when the system goes into $\mathbf{Q}_{\mathsf{tr}}$ its dynamics is stagnated. This mimics vain actions of an operator in directing the system motion towards the point $\mathbf{r}_{\mathsf{st}}$ precisely. Indeed, when the system under the operator control gets any point in $\mathbf{Q}_{\mathsf{tr}}$ the operator may consider the current situation perfect because he just does not "see" $\mathbf{r}_{\mathsf{st}}$ and until the system leaves $\mathbf{Q}_{\mathsf{tr}}$ he has no reason to keep the control active. The goal of the present work is to demonstrate that the fuzzy rationality can be responsible for complex emergent phenomena in such systems.

2 Lazy bead model

The following model captures the basic features of such human behavior. Let us consider a chain of N "lazy" beads (Fig. 1). Each of these beads can move in the vertical direction and its dynamics is described in terms of the deviation $x_i(t)$ from the equilibrium position and the motion velocity $v_i(t) = dx_i/dt$ depending on time t, here the bead index i runs from 1 to N. The equilibrium position $x_i = 0$ is specified assuming the formal initial ($i = 0$) and terminal ($i = N + 1$) beads to be fixed. Each bead i "wishes" to get the "optimal" middle position with respect to its nearest neighbors. So one of the stimuli for it to accelerate or decelerate is the difference

$$\eta_i = x_i - \frac{1}{2}(x_{i-1} + x_{i+1})$$

provided its relative velocity

$$\vartheta_i = v_i - \frac{1}{2}(v_{i-1} + v_{i+1})$$

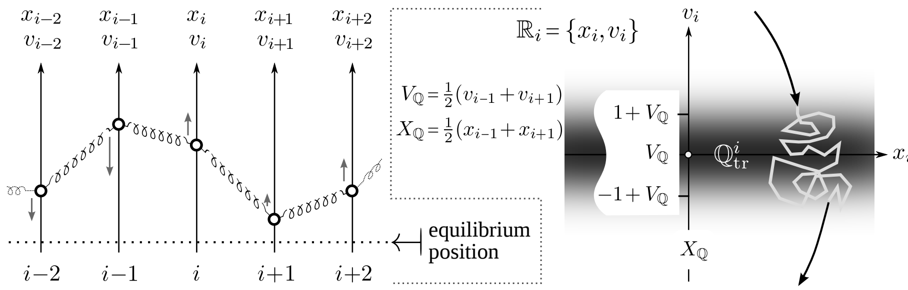

Fig. 1. The chain of N beads under consideration and the structure of their individual phase space $\mathbf{R}_i = \{x_i, v_i\}$ $(i = 1, 2, \ldots, N)$. The formal initial $i = 0$ and terminal $i = N + 1$ beads are assumed to be fixed, specifying the equilibrium bead position.

with respect to the pair of the nearest beads is sufficiently low. Otherwise, especially if bead i is currently located near the optimal position, it has to eliminate the relative velocity ϑ_i, representing the other stimulus for bead i to change its state of motion. The model to be formulated below combines both of these stimuli within one cumulative impetus $\propto (\eta_i + \sigma\vartheta_i)$, where σ is the relative weight of the second stimulus.

When, however, the relative velocity ϑ_i becomes less then a threshold θ, i.e., $|\vartheta_i| \lesssim \theta$, bead i is not able to recognize its motion with respect to the nearest neighbors. Since a bead cannot "predict" the dynamics of its neighbors, it has to regard them as moving uniformly with the current velocities. So from its standpoint, under such conditions the current situation cannot become worse, at least, rather fast. In this case bead i just "allows" itself to do nothing, i.e., not to change the state of motion and to retard the correction of its relative position. This feature is the reason why such beads are called "lazy". Below we will use dimensionless units in which, in particular, the perception threshold is equal to unity $\theta = 1$.

Under these conditions the equation governing the system dynamics is written in the following form

$$\frac{dv_i}{dt} = -\Omega(\vartheta_i)[\eta_i + \sigma\vartheta_i + \sigma_0 v_i] + \epsilon\xi_i(t)\,. \tag{1}$$

If the cofactor $\Omega(\vartheta_i)$ were equal to unity, the given system would be no more then a chain of beads connected by elastic springs characterized by the friction coefficient σ. The term $\sigma_0 v_i$ with the coefficient $\sigma_0 \ll 1$ that can be treated as a certain viscous friction of the beads moving via a medium into which the given system is embedded has been introduced to prevent the beads from attaining extremely high velocities. The factor $\Omega(\vartheta_i)$ is due to the effect of dynamical traps and the ansatz

$$\Omega(\vartheta) = \frac{\Delta + \vartheta^2}{1 + \vartheta^2}\,, \tag{2}$$

is used, where the parameter $\Delta \in [0, 1]$ quantifies the intensity of dynamical traps. If $\triangle = 1$, the dynamical traps do not exist at all, in the opposite case,

$\triangle \ll 1$, their influence is pronounced inside the neighborhood $\mathbf{Q}^i_{\mathsf{tr}}$ of the axis $v_i = (v_{i-1} + v_{i+1})/2$ (the trap region) whose thickness is about unity (Fig. 1). Model (1) allows for random factors in terms of white noise $\xi_i(t)$ affecting the motion of bead i with intensity ϵ so that

$$\langle \xi_i(t) \rangle = 0 \quad \text{and} \quad \langle \xi_i(t)\xi_{i'}(t') \rangle = \delta_{ii'}\delta(t - t')\,. \tag{3}$$

For the terminal fixed beads, $i = 0$ and $i = N + 1$, we set

$$x_0(t) = 0\,, \qquad x_{N+1}(t) = 0\,, \tag{4}$$

which play the role of the "boundary" conditions for equation (1).

It should be noted that the emergent phenomena in a similar system mimicking car following dynamics were considered for the first time in Refs [13,14]. In addition, the first experimental evidence of the dynamical traps caused by the human fuzzy rationality seems to be obtained in hybrid human-computer experiments of balancing a damped virtual stick [15].

3 Results of simulation

The dynamics of the given system was studied numerically. Initially all the beads were located at the equilibrium positions $\{x_i|_{t=0} = 0\}$ and perturbations were introduced into the system via ascribing random independent values to their velocities. Equation (1) was integrated using the E2 high order stochastic Runge-Kutta method [16]. The integration time step of 0.001 was used; the obtained results were checked to be stable with respect to decreasing the integration time step tenfold. The integration time was equal to 10^5–10^6, which enabled us to deal with the steady state dynamics. The other parameters used in simulation were taken equal to $\mathit{\Delta} = 10^{-3}$ and $\sigma_0 = 0.01$. Besides, to simply the data visualization the bead coordinates are shown with some individual shifts, namely, $x_i \to x_i + 50 \cdot i$.

In order to analyze the dynamical trap effect on its own the noise absence case was studied first. The system dynamics was found to depend on the intensity of "dissipation" quantified by the parameter σ. We remind that the parameter σ specifies the relative weight of the stimuli to take the middle "optimal" position and to eliminate the relative velocity; the larger the parameter σ, the more significant the latter stimulus. When the parameter σ is not too small the system tends to get the regime of regular dynamics represented by a collection of limit cycles of individual bead motion. It should be noted that these limit cycles could be of complex form when the number of beads is not too large, namely, $N \lesssim 10$ [17]. Nevertheless for systems with large number of beads the resulting phase portrait takes a rather universal form shown in Fig. 2(left frame). However, the "time to formation" T_N, i.e. the mean time required for a given bead chain to get the steady state regular dynamics grows exponentially as the number of beads increases. For example, for beads with $\sigma = 1$ this time can be approximated by the function

$$T_N \approx T_c \cdot \exp\{N/N_c\} \qquad \text{with } T_c \sim 60 \text{ and } N_c \sim 13 \tag{5}$$

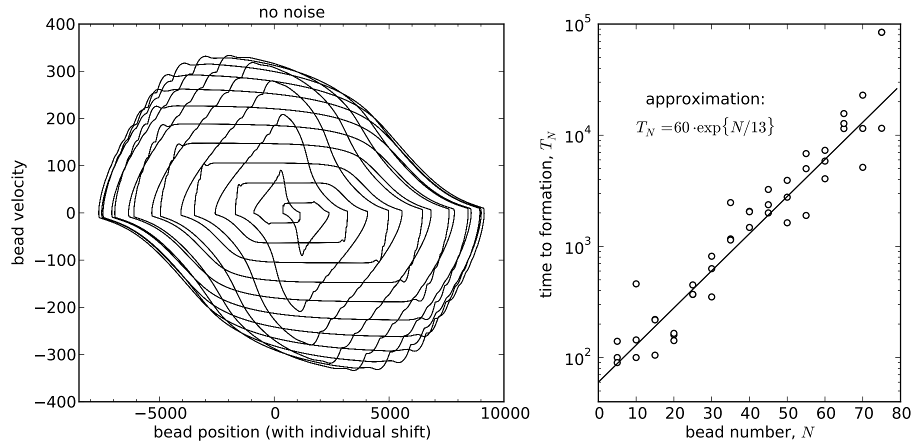

Fig. 2. The characteristic phase portrait of the steady state dynamics exhibited by systems without noise and not too weak "dissipation" (left frame). The chain of 30 beads with $\sigma = 1$ was used in constructing the shown pattern where the limit cycles of each second bead are visualized. The right frame depicts the characteristic time T_N required for such a system to get the steady state dynamics vs the number N of beads. The scatted points are the data obtained for each value of N on three trials, $\sigma = 1$ was used in simulation.

(see Fig. 2(right frame)). On one hand, this strong dependence explains that for chains of oscillators with not too weak "dissipation" only chaotic motion was found when the number of beads becomes sufficiently large, $N \gtrsim 100$ [17]. On the other hand, it enables us to pose a question about regarding the chaotic dynamics of such systems for $N \to \infty$ as a certain phase state.

In the case of weak "dissipation" the system dynamics exhibits sharp transition to a stable chaotic regime as the coefficient σ decreases. It is demonstrated in Fig. 3 showing the transition from the regular dynamics for $\sigma = 0.1$ to a chaotic motion when $\sigma = 0.09$. As seen in Fig. 3 the chaotic portrait can be conceived of as a highly chaotic kernel surrounded by fragments of the regular limit cycle destroyed by instability.

Noise forces these systems to undergo two phase transitions as its intensity ϵ increases. The first one can be categorized as the transition from the regular bead motion to a cooperative chaotic bead motion. The latter means that the beads correlate substantially with one another in motion but individual trajectories are rather irregular and the magnitude of this irregularity cannot be due to the present noise only. The second transition is determined by the formation of highly irregular mutually independent oscillations in the bead position. To illustrate the first phase transition Figure 4 depicts two phase portraits of the middle bead motion for different values of ϵ. As seen, for $\epsilon = 0.01$ the phase portrait looks like a regular limit cycle disturbed by small noise. In contrast, when the noise intensity increases by two times, i.e., $\epsilon = 0.02$, the corresponding phase portrait becomes rather complex in form and the volume of the phase space layer containing the shown trajectory as a whole

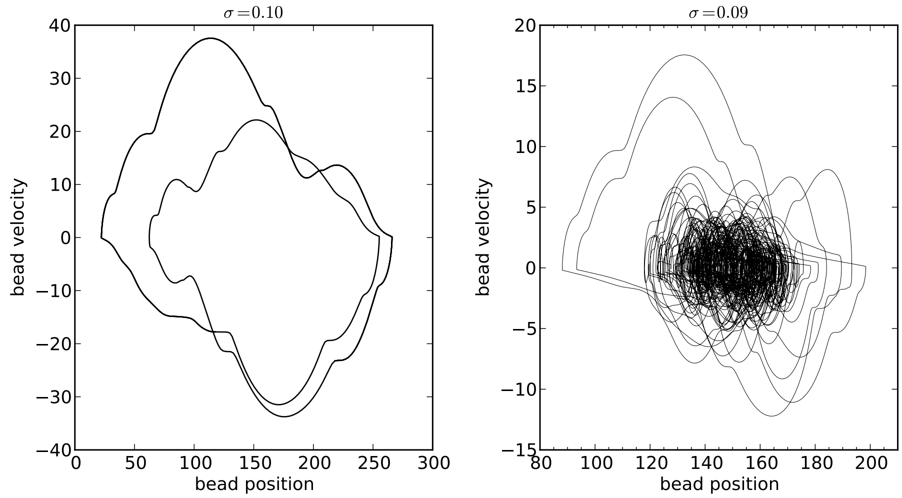

Fig. 3. The phase portraits of the middle bead motion of the 5-bead chain for the "dissipation" parameter σ taking the values 0.1 (left frame) and 0.09 (right frame). The period of the shown limit cycle is about 200; the chaotic phase portrait was obtained by visualizing the system motion within time interval about 5×10^5.

sharply grows. Exactly the two features has enabled us to classify the found effect as a phase transitions. It should be noted, that this phase transition from regular motion to stochastic chaos, in contrast to the second transition to highly irregular motion, does not manifest itself in the one-particle distributions of all the variables x, v, η, ϑ ascribed to the beads individually, so, it could be categorized as a "weak" phase transition.

4 Conclusion

The notion of dynamical traps was introduced to describe possible effects caused by the bounded capacity of human cognition in ordering events or actions according to their preference. Its particular implementation is that human beings as active elements of a certain system cannot individually control all the governing parameters within the accuracy required for stabilizing the system dynamics perfectly. Therefore one chooses a few crucial parameters and mainly focuses attention on them. When the equilibrium with respect to these crucial parameters is attained the human activity slows down, retarding in turn the system dynamics as a whole.

By way of example, we considered emergent phenomena in chains of coupled oscillators with dynamical traps. The motion of oscillating particles (beads) in the phase space $\{x_i, v_i = \dot{x}_i\}$ is assumed to be governed by their interaction via effective elastic springs with viscous friction outside the dynamical trap region $\mathbf{Q}_{\mathsf{tr}}$. For a given bead i the dynamical trap effect is reduced to depressing its interaction with the nearest neighbors $i-1$ and $i+1$ as the relative velocity

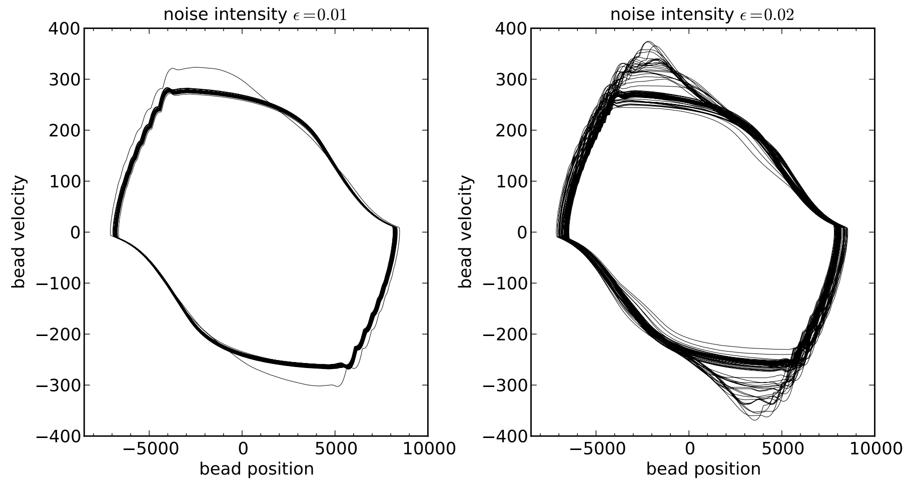

Fig. 4. The phase portraits of the middle bead motion of the 30-bead chain with $\sigma = 1$ for two values of the noise intensity $\epsilon = 0.01$ and 0.02. In plotting these portraits bead trajectories of motion during time interval about 2×10^4 were used.

$\vartheta_i = v_i - (v_{i-1} + v_{i+1})/2$ becomes small in comparison with some threshold. The introduction of additive white noise of intensity ϵ allows for possible uncontrollable factors also affecting the bead motion.

This system was studied numerically. As demonstrated, without noise the system dynamics tends to the regime of regular bead motion if the friction coefficient is not too small. However, the characteristic time required for a given system to get this regime grows exponentially with the number N of beads. It enables us to pose a question about regarding the chaotic transient processes as a certain phase state in the limit $N \to \infty$. When the friction coefficient becomes sufficiently small the steady state dynamics of such systems can undergo transition to chaotic bead motion even for chains with small number of beads. Depending on its intensity noise can induce the formation of three characteristic phases, highly irregular individual oscillations of the beads, the cooperative chaotic bead motion, and the synchronized regular bead motion. It should be noted that the transition between the regimes of regular and cooperative chaotic bead motion manifests itself only the sharp growth of the volume of the phase space layer containing the bead trajectories, whereas all the one-particle distribution functions does not change their forms remarkably.

Acknowledgments

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