Chaotic Modeling and Simulation (CMSIM) 1: 83-88, 2013

Poincaré Analysis of Non-linear Electromagnetic Modes in Electron-Positron Plasmas

V. Saxena and I. Kourakis

Centre for Plasma Physics, School of Mathematics and Physics,
Queen's University Belfast BT7 1NN , Northern Ireland, UK

Email : v.saxena@qub.ac.uk

Abstract : We present an investigation of coupled nonlinear electromagnetic modes in an electron-positron plasma by using the well established technique of Poincaré surface of section plots. A variety of nonlinear solutions corresponding to interesting coupled electrostatic-electromagnetic modes sustainable in electron-positron plasmas is shown on the Poincaré section. A special class of localized solitary wave solution is identified along a separatrix curve and its importance in the context of electromagnetic wave propagation in an electron-positron plasma is discussed.
Keywords : Poincaré section, solitary, electron-positron plasma

1.Introduction

The method of Poincaré surface of section (SOS) plots has been very useful in analysing higher dimensional non-linear dynamical systems [1]. For a given n-dimensional continuous dynamical system, the corresponding Poincaré SOS plot represents an equivalent discrete dynamical system with (n-1) dimensions and thus facilitates the analysis of possible periodic, quasi-periodic and chaotic modes, the original system can sustain. As non-linearity in plasmas is inherent they provide a perfect paradigm to study various non-linear processes ranging from coherent solitary waves to chaos and turbulence. In this respect, the subject of intense laser plasma interactions has ever received a great deal of attention. There has recently been a resurgence in this research area after the efficient production of very intense laser pulses ($I \geq 10^{18} W/cm^2$) has become a reality [2]. Laser pulses with such high intensities are called relativistically intense as the associated transverse electric fields are strong enough to drive the electrons to relativistic speeds. From theoretical point of view, these high intensity laser plasma interactions provide a favourable environment for a whole range of non-linear processes. Among them the formation of electromagnetic

Received: 5 February 2012 / Accepted: 12 September 2012

ISSN 2241-0503

solitary wave is a topic of much fundamental interest particularly in theoretical plasma physics. There have been several theoretical investigations addressing the existence and stability of coupled electromagnetic solitary waves in plasmas [3].

On the other hand, the electron-positron plasmas are thought to be a constituent of various astrophysical environments e.g. in pulsar magneto-spheres, in bi-polar flows in active galactic nuclei (AGN) and at the centre of our galaxy and are believed to be the first state of matter in the early stage of universe [4,5]. The coupling of electromagnetic waves to electron-positron plasmas is therefore an active area of theoretical research and has been addressed in few earlier works [5]. We present here a detailed Poincaré section based analysis of a class of possible coupled non-linear electromagnetic modes in an un-magnetized electron positron plasma with a particular emphasis on the coupled solitary waves solutions. This work is an extension of earlier works by Saxena et al. [7] and O. B. Shiryaev [6]. We adopt the same formalism as used by Kaw et al. [8] for an electron plasma with ions forming a neutralizing background.

2.Mathematical Model

The coupling of a relativistically intense electromagnetic wave with an electron-positron plasma is described by the following set of coupled fluid-Maxwell equations.

$$A_{xx} - A_{tt} = \left(\frac{n_e}{\gamma_e} + \frac{n_p}{\gamma_p} \right) A \tag{1}$$

$$\phi_{xx} = n_e - n_p \tag{2}$$

$$\left(p_{e,p}\right)_t = \left(\square\phi - \gamma_{e,p}\right)_x \tag{3}$$

$$\left(n_{e,p}\right)_t + \left(\frac{n_{e,p}\, p_{e,p}}{\gamma_{e,p}} \right)_x = 0. \tag{4}$$

Here indices e and p stand for electron and positron species respectively, A, Φ, $n_{e/p}$, and $p_{e/p}$ respectively represent the electromagnetic vector potential, electrostatic potential, the electron/positron density and electron/positron longitudinal momentum. $\gamma_{e,p}$ Is the relativistic factor given by,

$$\gamma_{e,p} = \sqrt{1 + |A|^2 + p_{e,p}^2} \tag{5}$$

By performing a co-ordinate transformation defined as $\xi = x - \beta t$ where $\beta = v_{ph} / c$ is the normalized phase velocity, one obtains following set of coupled non-linear ordinary differential equations.

$$\left(\beta^2 - 1\right)a_{\xi\xi} + \left(\frac{\beta}{\sqrt{\left(\beta^2 - 1\right)\left(1 + a^2\right) + \left(1 + \phi\right)^2}} + \frac{\beta}{\sqrt{\left(\beta^2 - 1\right)\left(1 + a^2\right) + \left(1 - \phi\right)^2}}\right)a = 0 \quad (6)$$

and

$$\phi_{\xi\xi} + \frac{1}{\left(\beta^2 - 1\right)}\left(\frac{\beta\left(1 + \phi\right)}{\sqrt{\left(\beta^2 - 1\right)\left(1 + a^2\right) + \left(1 + \phi\right)^2}} + \frac{\beta\left(1 - \phi\right)}{\sqrt{\left(\beta^2 - 1\right)\left(1 + a^2\right) + \left(1 - \phi\right)^2}}\right) = 0 \quad (7)$$

Now making a change of variable defined by

$$\left(\beta^2 - 1\right)^{1/2} a = X$$

$$1 + \phi = -Z$$

$$\frac{\xi}{\left(\beta^2 - 1\right)^{1/2}} = \xi'$$

we get following set of simplified equations,

$$\ddot{X} + \beta\left[\frac{1}{\sqrt{\beta^2 - 1 + X^2 + Z^2}} + \frac{1}{\sqrt{\beta^2 - 1 + X^2 + \left(Z + 2\right)^2}}\right]X = 0 \quad (8)$$

$$\ddot{Z} + \beta\left[\frac{Z}{\sqrt{\beta^2 - 1 + X^2 + Z^2}} + \frac{\left(Z + 2\right)}{\sqrt{\beta^2 - 1 + X^2 + \left(Z + 2\right)^2}}\right] = 0 \quad (9)$$

Above coupled equations (8) and (9) admit following constant of motion:

$$H = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\dot{Z}^2 + \beta\left[\sqrt{\beta^2 - 1 + X^2 + Z^2} + \sqrt{\beta^2 - 1 + X^2 + \left(Z + 2\right)^2}\right] \quad (10)$$

This problem is similar to that of coupled oscillators in Hamiltonian mechanics with two degrees of freedom and we solve above set of equations (8-10) using Runge-Kutta 4th order integration method to obtain coupled non-linear solutions.

3. Non-linear Solutions on Poincare Surface of Section

We consider the case of $\beta > 1$ and show the possible solutions on a Poincaré SOS plot defined by $X = 0, \dot{X} > 0$. We have investigated two interesting regimes $\beta - 1 << 1$ and $\beta - 1 \leq 1$. The results are shown in Fig.1 and Fig.2 respectively. It is worth noting that in the regime of phase velocities close to the speed of light, there exist a more varied class of solutions. The Poincaré plot in Fig.1 is obtained for $\beta = 1.001; H = 10$. The densely filled curves correspond to quasi periodic solutions with the ratio of the frequencies of two oscillators being a prime number. The centres of the left and right halves of the Poincaré plot represent the fixed points of zero measure and correspond to periodic orbits. The interesting island curves correspond to amplitude modulated quasi periodic modes whereas centres of these islands represent the fixed points of higher orders and correspond to periodic waves with an integer ratio of the two oscillator's frequencies. We note that the separatrix curve is not quite periodic and therefore indicates a possibility of slightly chaotic solutions.

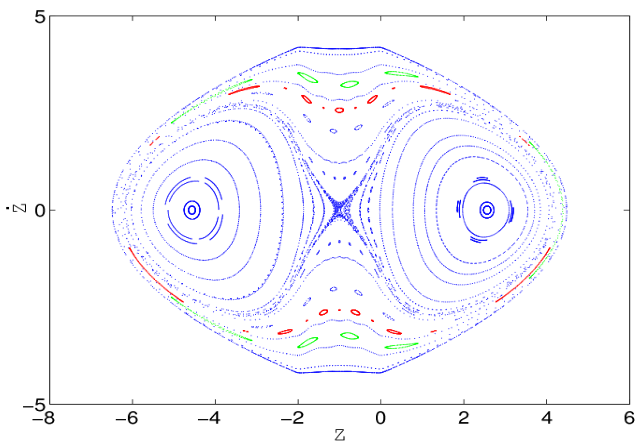

Fig.1 : Poincaré section plot for parameters β=1.001, H=10.

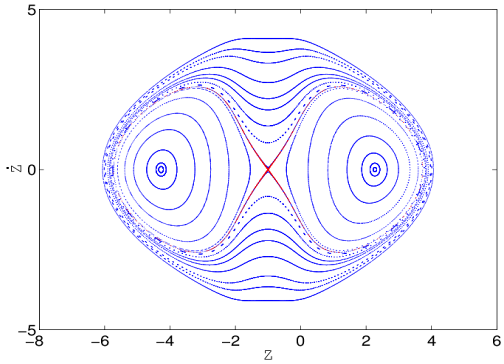

Fig.2 : Poincaré section plot for parameters β=1.1, H=10.

Now in the regime of $\beta - 1 \leq 1$, we choose the parameters to be $\beta = 1.1; H = 10$. The Poincaré surface of section plot for this case is shown in Fig.2. In this case we observe that the small island curves cease to exist. Moreover, there exists a sharp separatrix curve. This separatrix curve corresponds to localized solitary wave solutions. We show this particular solution in Fig.3.

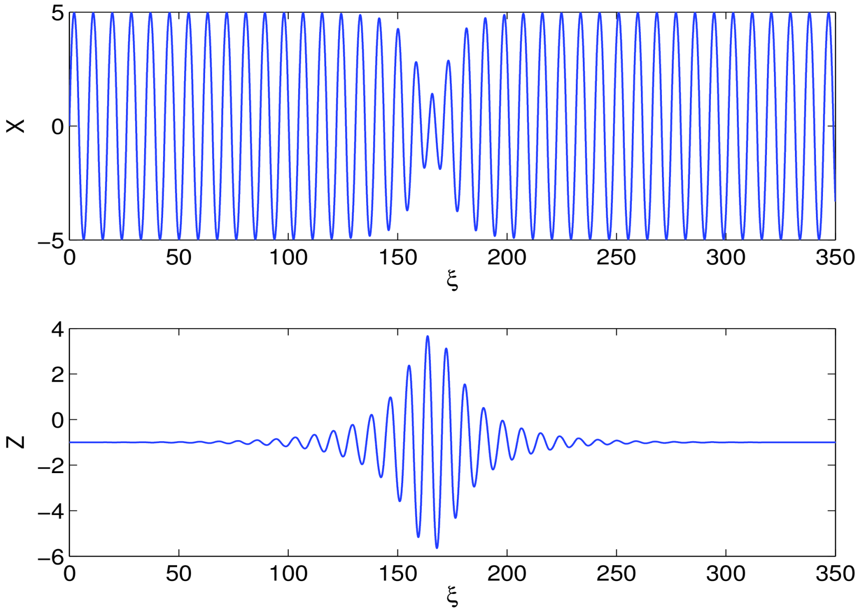

Fig.3 : Solitary solution corresponding to the separatrix curve in Fig.2.

4. Conclusions

To conclude, we have presented a class of coupled non-linear electromagnetic solutions for electromagnetic wave propagation in an electron-positron plasma by using Poincaré surface of section technique. A special class of solitary wave solutions has been identified along the separatrix curve in a parameter regime with phase velocities exceeding the speed of light by ~ 10% or more. These solitary modes play an important role in the energy localization in laser plasma interactions and therefore their stability needs to be understood which is an open area of research.

5. Acknowledgements

The authors gratefully acknowledge funding from the UK EPSRC (Engineering and Physical Science Research Council) via grant EP/I031766/1 and also (IK) via grant EP/D06337X/1. The figures are reproduced from *Europhysics Letters*, 100 15002 (2012) (doi: 10.1209/0295-5075/100/15002).

References

1. H. Poincaré 1892 *Les methods nouvelles de la mechanique celeste* (Paris: Gauthier-Villars)
2. G. Mourou and D. Umstadter, Phys. Fluids B, vol.4, 2315 (1992).
3. V. Saxena et al., Phys. Plasmas, vol. 14, 072307 (2007) *and references therein*.
4. F.C. Michel, Rev Mod. Phys., vol. 54,1 (1982).
5. M.C. Begelman et al., Rev. Mod. Phys., vol 56, 255 (1984).
6. O.B. Shiryaev, phys. Plasmas, vol 13, 112304 (2006) *and references therein.*
7. V. Saxena, A. Sen, and P. Kaw, Phys. Rev. E, vol 80, 016406 (2009).
8. P.K. Kaw, A. Sen, and E.J. Valeo, Physica D, vol 9, 96 (1983).