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The Energy of Generalized Logistic Maps at Full Chaos

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Abstract: The energy of generalized logistic maps at full chaos is computed and examined in serving as a guide to multi-disciplinary applications. The maps considered are of the form $x_{n+1} = f(x_n) = r\, x_n^{\lambda}(1-x_n)^{\mu}$, limited above by 1 resulting in the maximum r that yields full chaos; the exponents λ and μ are taken to be any positive real numbers between 0.5 and 2. For given values of r, λ and μ, the average energy is calculated as the average squared x over 512 points starting at the 2049th iteration point. Near full chaos, its dependence on r for fixed values of λ and μ is highly non-linear consisting of a number of maxima and minima. For λ greater than 1.1, the energy diminishes independent of the initial iteration point. At full chaos, the energy dependence on values of λ in the range [0.5, 1.1] and values of μ in the range [0.5, 2] is depicted graphically. For a fixed λ or μ, this dependence is approximated linearly.
Keywords: energy, full chaos, logistic map**,** generalized logistic maps.

1 Introduction

In the present paper, the energy of chaotic generalized logistic maps is computed and examined. The results of the present study may find applications in diverse scientific disciplines such as fracture mechanics, see for example, D. Sotiropoulos [1], social sciences (e.g. Skiadas & Skiadas [2]), population growth modeling (e.g. Marotto [3]), and music composition (e.g. D. Sotiropoulos et al [4]). For applications in astronomy and other areas the reader is referred to the monograph of Skiadas & Skiadas [5].

The generalized logistic maps considered here are of the form,

$$x_{n+1} = r x_n^{\lambda} (1 - x_n)^{\mu} \tag{1}$$

in which the parameters r, λ and μ are positive real numbers, while the variable x and its map range from 0 to 1. The classical logistic map is given by λ=1 and μ=1, whose chaotic nature of the produced x's were discussed by May [6]. A discussion on the x's produced by iteration for λ=1 and μ<1 as well as other specific values may be found in Skiadas & Skiadas [5]. Marotto [3] found that for the case λ=2 and μ=1, there is a range of values of r near its maximum (which is obtained from the condition that the upper limit of x is 1) for which the

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x's produced are soon after attracted to zero, independent of the initial x chosen in the iteration process. Gottlieb [7] further discussed this by computing a region in the (initial x, r) space in which the produced x's escape the zero value fixed point. The chaotic behavior of the case $\lambda=\mu=1/2$ as applied to music composition was visited by V. Sotiropoulos [8], while that of the double logistic map was examined for music composition by by A. Sotiropoulos [9]. On maps of related functional dependence, Stutzer [10] investigated the iterative map $x_{n+1} = r\ x_n\ (1-x_n^{1/2})$ as a macro-economic dynamic model, while Gottlieb [11] analyzed the map $x_{n+1} = r\ x_n^{3/2}\ (1-x_n^{1/2})$. Skiadas & Skiadas [5] looked at the chaotic behavior of the generalized rational iteration model $x_{n+1} = x_n + r\ x_n\ (1-x_n)/[1-(1-\sigma)x_n]$ with positive σ. Last, D. Sotiropoulos [12] examined in detail the nature and regions of existence of fixed points for the map given by Eq,. (1) above.

The upper limiting value of the map parameter r in Eq. (1) is given in D. Sotiropoulos [12] in terms of the map exponents λ and μ as

$$r_{\max} = (1+\lambda/\mu)^{\mu}(1+\mu/\lambda)^{\lambda} \qquad (2)$$

It is at this value of r which yields full chaos that the energy produced by the map of Eq. (1) will be calculated and studied in the present paper for values of λ in the range [0.5, 1.1] and values of μ in the range [0.5, 2], since it is found in the present study that for λ greater than 1.1 the energy diminishes independent of the initial iteration point. Furthermore, for a fixed λ (or μ) the dependence of the map's energy at full chaos on μ (or λ) will be established analytically by approximating the energy computed from x's resulting from the iterations of Eq. (1).

2 The energy of generalized logistic maps

The energy, E, of the generalized logistic maps given by Eq. (1) is defined as the sum of the squared iterated x's

$$\mathrm{E} = \sum_{n=1}^{N} x_n^2 \qquad (3)$$

The energy clearly depends not only on the map's parameters r, λ and μ but also on the number, N, of x's taken into account in the summation and on the initial x ($=x_1$) selected. To eliminate the energy's dependence on the latter two factors, we divide the energy by a large number N and pick as initial x, the x produced by the map after a large number of iterations so that the resulting energy will be independent of both N and the initial x. Thus, we define the average energy which may be interpreted as the map's generated power as

$$\overline{\mathrm{E}} = \mathrm{E}/\mathrm{N} \qquad (4)$$

The average energy being, therefore, the map's average squared iterated x.

From the numerical calculations performed in the present study, we have concluded that the appropriate initial x to choose in order to satisfy the aforementioned requirement is the x produced by the map after 2049 iterations. Moreover, in computing an invariant value for the average energy a large number of x's needs to be taken into account and we have concluded that 512 iterations are enough to satisfy this requirement.

As an example of the chaos generated by the map of Eq. (1), the 512 chaotic x's generated at full chaos ($r=r_{max}$) after 2049 iterations with $\lambda=1$ and $\mu=0.5$, 1, 2 are shown in Fig. 1a, b, c.

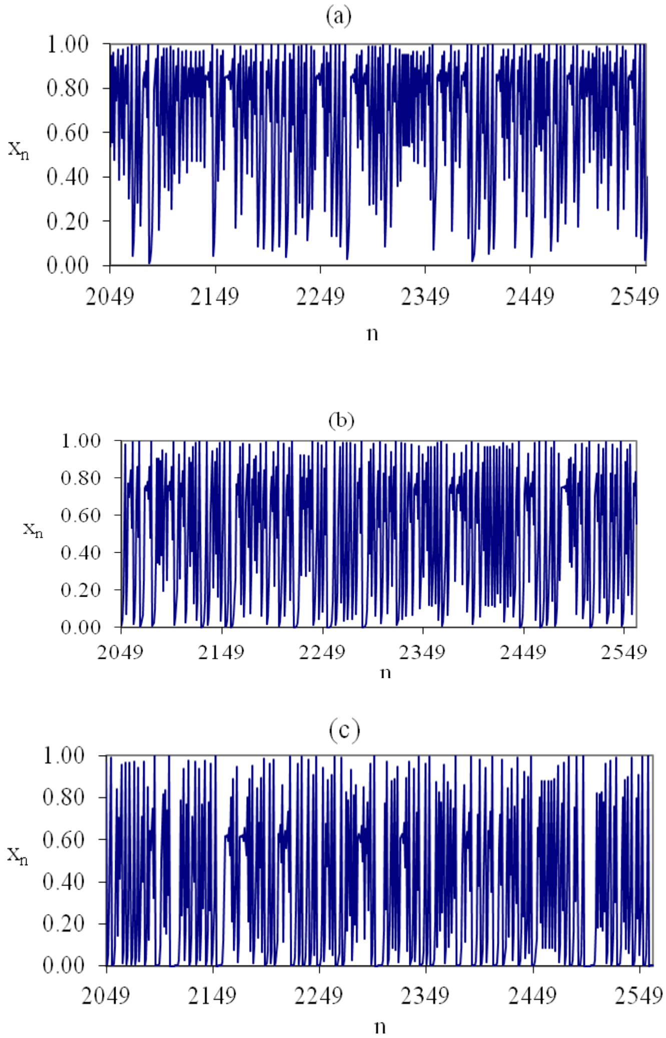

Fig. 1. The fully chaotic productions, x_n, of the generalized logistic maps with λ=1and μ=0.5 (α), 1 (b), 2 (c)

In Fig.1, the case $\lambda=\mu=1$ ($r=4$) corresponds to the classical logistic map [6]. We see that the generated fully chaotic x's are different for the three cases (a) (b) and (c) and, as we shall see in the following section, also the average energy produced is different.

Next, the average energy, $\bar{E}$, of the generalized logistic maps of Eq. (1) is computed using Eqs. (3), (4) for a large range of the map's multiplying parameter r and for different values of λ and μ. The upper limiting value of λ is taken as 1.1 since we have found computationally that larger values yield diminishing energy near full chaos (maximum r) as the generated x's go to the fixed point zero after only very few iterations. The phenomenon of diminishing generated x's near full chaos for the map of Eq. (1) with $\lambda=2$ and $\mu=1$ was observed and explained by Marotto [3] and further discussed by Gottlieb [7] in terms of the nature of fixed points for this value of λ. In view of the findings by the second author of the present paper in [12] in respect of the nature and existence of fixed points for all values of λ and μ in the map of Eq. (1), we see that Marotto's explanation holds true for diminishing x's near full chaos for λ's greater than 1.1.

The computed average energy, $\bar{E}$, versus the generalized logistic map parameter r is shown in Fig. 2 for different values of equal map exponents, $\lambda=\mu$. Interest in the present study is in the chaotic regime or near it so that very small r's are not considered in the computations. We observe that the maximum r considered increases with increasing $\lambda=\mu$ in accordance with Eq. (2). Further, we observe that the average energy decreases both at its maximum and at full chaos (maximum r) with increasing $\lambda=\mu$. Last, the highly non-linear dependence of energy on r *near* full chaos is evident with the existence of a number of extrema.

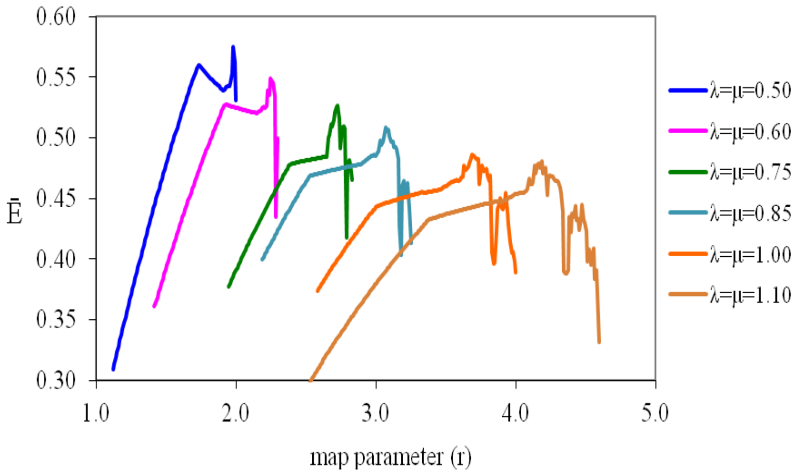

Fig. 2. The average energy, $\bar{E}$, versus the generalized logistic map parameter r for equal exponents, $\lambda=\mu$.

3 The energy at full chaos

Of particular interest in the present paper is the energy generated by the generalized logistic maps of Eq (1) at full chaos, that is, when $r=r_{max}$ as given by Eq. (2). To this end, the average energy, $\bar{E}$, is computed at full chaos for different values of the map exponents λ and μ. In Fig. 3, the average energy, $\bar{E}$, is shown versus the map exponents λ=μ. In solid line is the computed value.

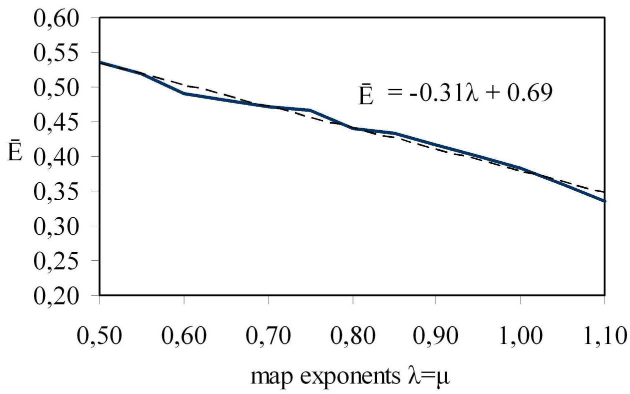

Fig. 3. The average energy, $\bar{E}$, at full chaos versus the map exponents, λ=μ.

We observe as already noted above, that the energy at full chaos decreases with increasing λ=μ. This decrease is substantial as exemplified by the relative energy decrease of 36% generated by the fully chaotic elliptic map (λ=μ=0.5) and the near-logistic map (λ=μ=1.1). Furthermore, we see that the decrease is weakly non-linear so that a linear approximation to the computed energy data points as performed by Excel results in the dashed line shown in the figure. The equation of the approximate linear dependence of the average energy on the map exponent λ (=μ) is also shown in the figure. The squared regression coefficient between the two is 0.99.

Next, the average energy, $\bar{E}$, at full chaos was computed versus one of the map exponents for fixed values of the other exponent. The results are shown in Figs. 4, 5. In Fig. 4 the energy dependence on the map exponent λ for fixed values of μ is shown.

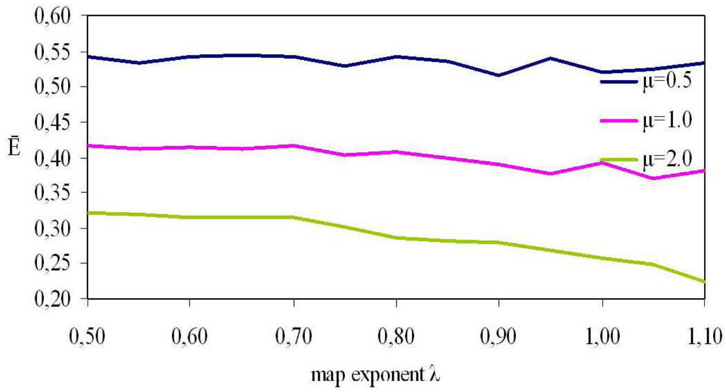

Fig. 4. The average energy Ē at full chaos versus the map exponent λ for fixed μ

It is observed that the average energy exhibits weak fluctuations with increasing λ and has a noticeable decrease with increasing λ only for the larger values of μ. To compare with this, Fig. 5 is shown where now λ is fixed and the energy dependence on the map exponent μ is depicted.

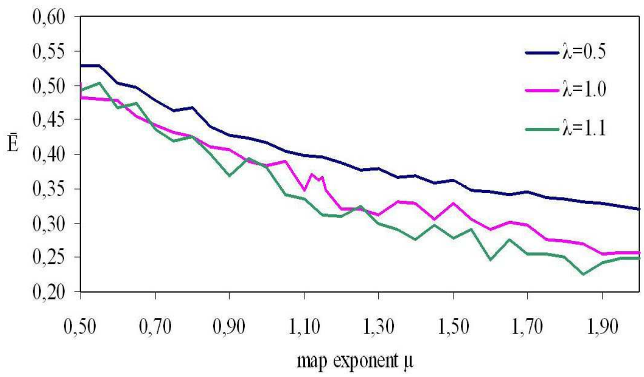

Fig. 5. The average energy Ē at full chaos versus the map exponent μ for fixed λ

It is seen that the energy's dependence on μ is stronger than on λ as far as its decrease is concerned.

Last, as demonstrated above for λ=μ, it will be shown in Figs. 6, 7 that the energy's dependence on λ or μ may also be linearly approximated for unequal λ and μ if one of the two exponents is kept constant.

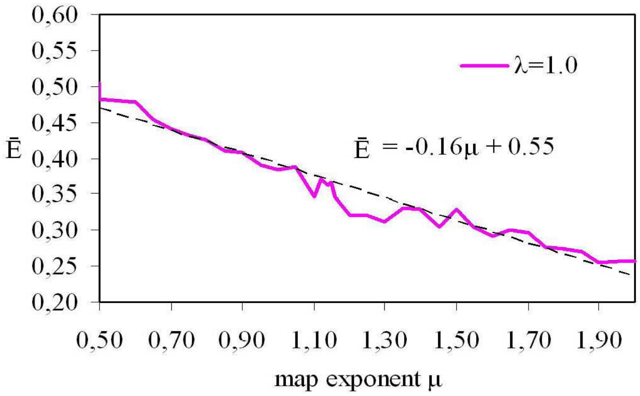

Fig. 6. The linear approximation of the average energy, Ē, at full chaos on μ.

In both Figs. 6, 7 the solid line connects the computed energy points at full chaos while the dashed line is the modeled linear approximation.

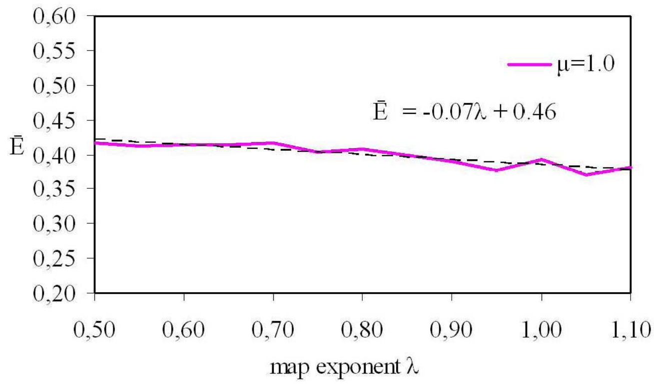

Fig. 7. The linear approximation of the average energy, Ē, at full chaos on λ.

The linear approximation of the energy with increasing map exponent λ or μ is satisfactory since the squared regression coefficients are 0.88 and 0.94 for Figs. 6 and 7, respectively.

4 Conclusions

The energy of generalized logistic maps was studied. In order to be able to compare the energy generated by different maps and also have an invariant energy value for each map, the average energy was defined for a large number (512) of iterations as the total energy per number of iterations with an initial map value that given by the map after a couple of thousand (2049) of iterations. It was found that the average energy exhibits strong fluctuations with a number of extrema near the chaotic regime whose full development is given by the maximum map parameter r.

Further, the energy at fully developed chaos decreases with increasing map exponents λ and μ and this decrease was satisfactorily approximated in a linear fashion.

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