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# Scale Invariant Model of Statistical Mechanics and Quantum Nature of Space, Time, and Dimension

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**Abstract.** Some of the implications of a scale invariant model of statistical mechanics to the physical and quantum nature of both space and time as well as quantum mechanics are described. At thermodynamic equilibrium, the velocity, energy, and speed of particles are shown to be governed by invariant Gaussian, Planck, and Maxwell-Boltzmann distribution functions. Physical space or Casimir vacuum is identified as a compressible tachyon fluid, Planck compressible ether, such that Lorentz-FitzGerald contractions become causal (Pauli) in accordance with Poincare-Lorentz *dynamic theory of relativity* as opposed to Einstein *kinematic theory of relativity*. Also, some of the implications of the model to the physical foundation of Riemann hypothesis are discussed. In particular, normalized spacing between non-trivial zeroes of Riemann zeta function are found to follow normalized Maxwell-Boltzmann distribution function.

## 1 Introduction

Similarities between stochastic quantum fields [1-17] and classical hydrodynamic fields [18-30] resulted in recent introduction of a scale-invariant model of statistical mechanics and its applications to thermodynamics [31, 32], fluid mechanics [33], and quantum mechanics [34-36].

In the present study, further implications of scale invariant model of statistical mechanics to the quantum nature of space and time and theory of relativity are investigated. Also, physical foundations of quantum mechanics as well as distribution of normalized spacing between non-trivial zeroes of Riemann zeta function are examined.

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## 2 A Scale Invariant Model of Statistical Mechanics

The scale-invariant model of statistical mechanics for equilibrium galactic-, planetary-, hydro-system-, fluid-element-, eddy-, cluster-, molecular-, atomic-, subatomic-, kromo-, and tachyon-dynamics corresponding to the scale β = g, p, h, f, e, c, m, a, s, k, and t is schematically shown on the left hand side of Fig. 1 [30]. For each statistical field, one defines particles that form the background fluid and are viewed as point-mass or "*atom*" of the field. Next, the *elements* of the field are defined as finite-sized composite entities composed of an ensemble

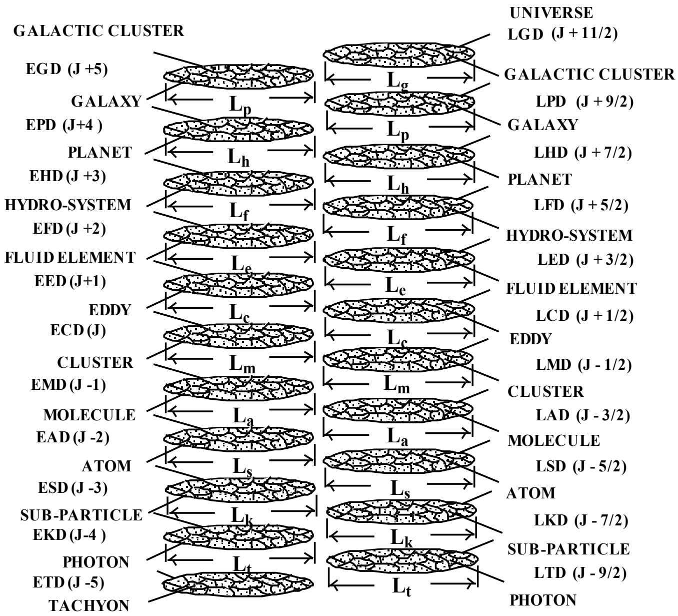


Fig. 1. A scale-invariant model of statistical mechanics. Equilibrium-β-Dynamics on the left-hand-side and non-equilibrium Laminar-β-Dynamics on the right-hand-side for scales β = g, p, h, f, e, c, m, a, s, k, and t as defined in Section 2. Characteristic lengths of (system, element, “atom”) are $(L_\beta, \lambda_\beta, \ell_\beta)$ and $\lambda_\beta$ is the mean-free-path [32].

an ensemble of "*atoms*". Finally, the ensemble of a large number of "*elements*" is defined as the statistical "*system*" at that particular scale. The most probable element of the lower scale β is identified as the “atom” $\mathbf{v}_{mp\beta} = \mathbf{u}_{\beta+1}$ of the next higher scale β + 1 leading to hierarchy of statistical fields shown in Fig. 2

## 3 Physical Space Identified as a Compressible Fluid that is Casimir Vacuum or Aristotle Fifth Element

A most significant implication of the model in Figs. 1 and 2 concerns the nature of physical space, Casimir [37] vacuum, that is identified as a *tachyonic fluid* that is de Broglie hidden thermostat [3]. It is emphasized that *space* is the *tachyonic fluid itself* and not *merely a container that is occupied by* this fluid, as in the classical theories of ether [38]. Using a glass of water as an example, the physical space is analogous to the water itself, and not to the glass.

Both Descartes and Huygens recognized the significant role of Aristotle ether in phenomena of gravitation and optics. For example, in harmony with the scale-invariant model of statistical mechanics shown in Fig. 1, the propagation of light in a medium called ether was suggested to be analogous to propagation of sound in air by Huygens [39]

*"As regards the different modes in which I have said the movement of Sound and of Light are communicated, one may sufficiently comprehend how this occurs in the case of Sound if one considers that the air is of such nature that it can be compressed and reduced to a much smaller space than that which it ordinarily occupies,"*

Also, the fundamental role of physical space, Casimir vacuum [37], in constitution of matter according to modern particle physics was anticipated by Leibniz who stated [40]

*"If space is an absolute reality, far from being a property or accident opposed to substance, it will have a greater reality than substances themselves,"*

Similar arguments concerning the important role of the ether as the seat of gravitational and electromagnetic phenomena were raised by Newton [41] and Maxwell [42]. The existence of the medium called *ether* was also found to be indispensable for the proper description of electrodynamics according to Lorentz [43, 44]

*"I cannot but regard the ether, which can be the seat of an electromagnetic field with its energy and its vibrations, as endowed with certain degree of substantiality, however different it may be from all ordinary matter,"*

The participation of ether in the transmission of perturbations as well as the possible granular structure of space were anticipated by Poincaré [45]

***"We might imagine for example, that it is the ether which is modified when it is in relative motion in reference to the material medium which it penetrates,***

***that when it is thus modified, it no longer transmits perturbations with the same velocity in every direction."***

Also, the notion of ether was considered by Einstein as not only consistent with the general theory of relativity, but in his opinion according to GTR space without ether is *unthinkable* [46]

*"Recapitulating, we may say that according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. According to the general theory of relativity space without ether is unthinkable; for in such space there not only would be no propagation of light, but also no possibility of existence for standards of space and time (measuring-rods and clocks), nor therefore any space-time interval in the physical sense. But this ether may not be thought of as endowed with the quality characteristic of ponderable media, as consisting of parts which may be tracked through time. The idea of motion may not be applied to it."*

The statement "space without ether" shows that ether was considered as a medium that *filled the space* rather than *being the space* itself. Also, because stochastic Planck and Boltzmann constants relate to *vacuum fluctuations* [32], contrary to the above statement by Einstein, the idea of *rest* rather than *motion* may not be applied to the ether. In other words, stochastic ether cannot satisfy both the principles of relativity and quantum mechanics if it is at rest. Ironically, parallel to static rather than dynamic vacuum at Planck scale, Einstein also chose a static rather than dynamic universe at cosmic scale (see Fig. 1) that resulted in his introduction of the cosmological constant.

After the development of quantum mechanics it was suggested by Dirac [47] that stochastic ether could satisfy both quantum mechanics and relativity theory

*"We can now see that we may very well have an aether, subject to quantum mechanics and conforming to relativity, provided we are willing to consider the perfect vacuum as an idealized state, not attainable in practice. From experimental point of view, there does not seem to be any objection to this. We must make some profound alterations in our theoretical ideas of the vacuum. It is no longer a trivial state, but needs elaborate mathematics for its description."*

When space is considered to be a *tachyonic compressible fluid* [36], Planck compressible ether [43], parallel to atmospheric air that becomes compressible when Mach number Ma = v/a approaches unity, the ether that constitutes the physical space becomes compressible when Michelson number defined as Mi = v/c approaches unity, with a and c denoting the velocity of sound and light respectively. Hence, changes of density when tachyon fluid is brought to rest isentropically will be given by [48]

$$\rho = \frac{M}{V} = \rho_o \left[1 + \frac{\gamma - 1}{2}\frac{v^2}{c^2}\right]^{\frac{1}{\gamma-1}} = \frac{M}{V_o}\left[1 + \frac{\gamma - 1}{2} Mi^2\right]^{\frac{1}{\gamma-1}} \quad (1)$$

The notation of subscript in Eq. (1) is opposite to that in conventional gas dynamics [49] where *stagnation* quantities correspond to *moving* fluid such as stagnation enthalpy $h_o = h + mv^2/2$. With $\gamma = 4/3$ for photon gas and assuming that the transverse coordinates do not change [50] for one dimensional compressible flow, Eq. (1) leads to Lorentz-FitzGerlad contraction [43, 32, 36]

$$V = V_o\sqrt{1 - v^2/c^2} \quad , \quad \ell = \ell_o\sqrt{1 - v^2/c^2} \quad , \quad \rho = \frac{\rho_o}{1 - v^2/c^2} \quad (2)$$

Thus, Ma >1 (Mi >1) corresponds to *supersonic* (*superchromatic*) flow, leading to formation of Mach (Poincaré-Minkowski) cone that separates the zone of *sound* (*light*) from the zone of *silence* (*darkness*) [48].

In view of the above considerations and in harmony with ideas of Darrigol [51] and Galison [52], one can identify two distinct paradigms of the special theory of relativity [36]:

**(A) Poincaré-Lorentz**
Dynamic Theory of Relativity
Space and time (x, t) are altered due to causal effects of motion on the ether.

**(B) Einstein**
Kinematic Theory of Relativity
Space and time (x, t) are altered due to the two postulates of relativity:

1- The laws of physics do not change form for all inertial frames of reference.
2- Velocity of light is a universal constant independent of the motion of its source.

According to dynamic theory of relativity the relativistic effects are causal as emphasized by Pauli [50] and induced by the dynamic effects of motion on the manifold of space, ether. It is also noted that strictly speaking, both postulates 1 and 2 above are not valid. This is because the speed of light is not a constant but a function of temperature of Casimir [37] vacuum and hence decreases ever so slowly with the expansion of the universe thus appears as constant on time scales relevant to human civilization. Also, the postulate of relativity is not valid since all inertial frames are distinguishable from one another through measurements with respect to stochastically stationary isotropic cosmic background radiation of Penzias-Wilson [53]. It appears that such distinguishability was known to Poincaré based on fundamental principles as suggested by the lecture he delivered in London in 1912 shortly before he died [54]

***"Today some physicists want to adopt a new convention. It is not that they are constrained to do so; they consider this new convention more convenient; that is all. And those who are not of this opinion can legitimately retain the old one in order not to disturb their old habits. I believe, just between us, that this is what they shall do for a long time to come,"***

## 4 A New Physical Foundation of Quantum Mechanics

According to a recent study [36], the energy spectrum of all isotropic equilibrium statistical fields shown in Fig. 1 will be governed by the invariant Planck energy distribution law [55, 36]

$$\frac{\varepsilon_\beta dN_\beta}{V} = \frac{8\pi h}{u_\beta^3} \frac{\nu_\beta^3}{e^{h\nu_\beta / kT} - 1} d\nu_\beta \tag{3}$$

when the energy of each oscillator is $\varepsilon_\beta = h\nu_\beta$. In amother recent investigation [56] the invariant Maxwell-Boltzmann distribution function

$$\frac{dN_{u\beta}}{N} = 4\pi \left(\frac{m_\beta}{2\pi k T_\beta}\right)^{3/2} u_\beta^2 \; e^{-m_\beta u_\beta^2 / 2kT_\beta} du_\beta \tag{4}$$

was directly derived from the invariant Planck energy distribution function. This result is to be expected since particle speeds are related to the square root of their kinetic energy. Hence, under thermodynamic equilibrium, the speed of particles will be governed by the invariant Maxwell-Boltzmann distribution function [56] leading to a hierarchy of embedded distributions at …ECD, EMD, and EAD … scales as shown in Fig. 2.

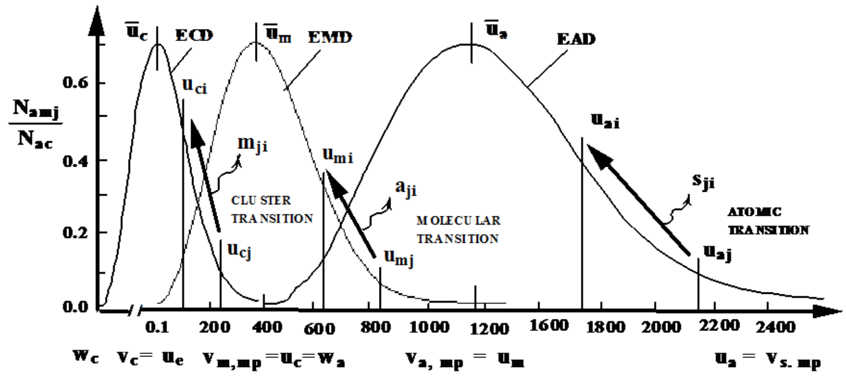

Fig. 2. Maxwell-Boltzmann speed distribution viewed as stationary spectra of cluster sizes for ECD, EMD, and EAD scales at 300 K [32].

According to Fig. 2, one may view Maxwell-Boltzmann distribution as distribution of stochastically stationary particle cluster sizes that could be also identified as different "energy levels" of quantum mechanics. Hence, one may introduce a new paradigm of the physical foundation of quantum mechanics according to which Bohr *stationary states* [57] correspond to the *stochastically stationary* sizes of particle clusters, de Broglie wave packets, governed by Maxwell-Boltzmann distribution function. For example, in iso-tropic turbulence corresponding to equilibrium eddy-dynamic at scale β = e, the statistically stationary sizes of "eddies" or "clusters of molecular-clusters" are governed by Maxwell-Boltzmann distribution function [36]. Hence, one views the transfer of a cluster from a small rapidly oscillating eddy j to a large slowly oscillating eddy i as transition from the high energy level j to the low energy level i (Fig. 2) as schematically shown in Fig. 3.

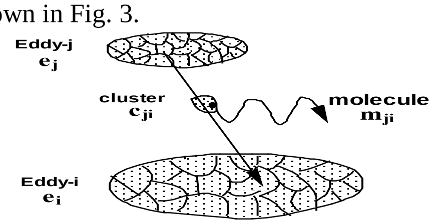

Fig. 3. Transition of cluster $c_{ij}$ from eddy-j to eddy-i leading to emission of molecule $m_{ij}$.

Such a transition will be accompanied with emission of a "molecule" that will carry away the excess energy $\Delta\varepsilon_{ji\beta} = \varepsilon_{j\beta} - \varepsilon_{i\beta} = h(\nu_{j\beta} - \nu_{i\beta})$ in harmony with Bohr [57] theory of atomic spectra. Therefore, the reason for the *quantum nature* of "molecular" energy spectra in equilibrium isotropic turbulent fields is that transitions must occur between eddies whose energy levels must satisfy the criterion of *stationarity* imposed by Maxwell-Boltzmann distribution function [36].

The invariant conservation equations for an incompressible and irrotational $\mathbf{v}_\beta = -\nabla\Phi_\beta$ flow with the velocity potential $\Phi_\beta$ lead to the invariant Bernoulli equation [36]

$$-\frac{\partial(\rho_\beta\Phi_\beta)}{\partial t_\beta} + \frac{(\nabla\rho_\beta\Phi_\beta)^2}{2\rho_\beta} + p_\beta = \text{constant} = 0 \tag{5}$$

Comparison of Eq. (5) with Hamilton-Jacobi equation [2] resulted in the introduction of the invariant action $S_\beta(\mathbf{x}, t) = -\rho_\beta\Phi_\beta$ and quantum mechanics wave function $\Psi_\beta(\mathbf{x}, t) = S'_\beta(\mathbf{x}, t) = -\rho_\beta\Phi'_\beta$ to derive from Eq. (5) the invariant time-dependent Schrödinger equation [58, 36]

$$i\hbar_{o\beta}\frac{\partial\Psi_\beta}{\partial t_\beta} + \frac{\hbar_\beta^2}{2m_\beta}\nabla^2\Psi_\beta - \overline{U}_\beta\Psi_\beta = 0 \tag{6}$$

It is therefore clear that the potential energy $\overline{U}_\beta$ in Eq. (6) like pressure in Eq. (5) acts as Poincaré *stress* [59-61] and is responsible for stability of “particles” or de Broglie wave packet [36]. Since in the absence of spin, potential flow, a Bernoulli equation (5) can be derived for each statistical field shown in Fig. 1, leading to a corresponding Schrödinger equation (6), the entire hierarchy of statistical fields from cosmic to photonic scales is governed by quantum mechanics. At cosmic scale the wave function $\Psi_g(\mathbf{x}, t)$ will correspond to Hartle-Hawking [62] wave function of the universe. The wave-particle duality of galaxies has been established by their observed quantized red shifts [63].

# 5 Distribution of Spacings between Zeroes of Riemann Zeta Function

The scale invariant model of statistical mechanics (Fig. 1) also impacts analytical number theory and hence Hilbert’s number eight problem namely Riemann hypothesis. Since Maxwell-Boltzmann speed distribution in Eq. (4) gives distribution of sizes of particle clusters, if expressed in dimensionless form it can also be viewed as distribution of sizes of “clusters of numbers” or Hilbert “condensations”. Therefore, a recent study [35] was focused on possible connections between the result in Eq. (4) and the theoretical findings of Montgomery [64] and Odlyzko [65] on analytical number theory and what is known as Montgomery-Odlyzko law [64-65]

*“The distribution of the spacings between successive non-trivial zeroes of the Riemann zeta function (suitably normalized) is statistically identical with the distribution of eigenvalue spacings in a GUE operator”*

The pair correlation of Montgomery [64] was subsequently recognized by Dyson to correspond to that between the energy levels of heavy elements [66-67] and thus to the pair correlations between eigenvalues of *Hermitian* matrices [68]. Hence, a connection was established between quantum mechanics on the one hand and quantum chaos [69] on the other hand. However, the exact nature of the connections between these seemingly diverse fields of quantum mechanics, random matrices, and Riemann hypothesis [66-67] is yet to be understood.

When particle speeds (cluster sizes) in Eq. (4) are normalized through division by the most probable speed (the most probable cluster size) one arrives at normalized Maxwell-Boltzmann (NMB) distribution function [35]

$$\rho_j = (8/\pi_\beta)\,[(2/\sqrt{\pi_\beta})x_{\beta j}]^2\ e^{-[(2/\sqrt{\pi_\beta})x_{\beta j}]^2} \qquad (7)$$

The additional division by the "measure" $\sqrt{\pi_\beta}/2$ in Eq. (7) is for coordinate normalization discussed in [35] and shown in Fig. 5. Direct comparisons between Eq. (7) and the normalized spacings between the zeroes of Riemann zeta function and the eigenvalues of GUE calculated by Odlyzko [65] are shown in Fig. 4. Therefore, a definite connection has been established between analytic number theory, the kinetic theory of ideal gas, and the normalized spacings between energy levels in quantum mechanics [35].

If one considers in the spirit of Pythagoras and Plato that pure "numbers" are the basis of all that is physically "real", then these "atomic" prime numbers applied to construct p-adic statistical field and their associated p-adic matrices [35] may lie at the foundation of Riemann hypothesis in harmony with noncommutative geometry [70]. That is, when the *physical space* or Casimir vacuum [37] is itself identified as a *fluid* governed by a statistical field [35, 36], it will have an energy spectrum given by Schrödinger equation (6) of quantum mechanics that in view of Heisenberg [71] *matrix mechanics* will be described by noncommutative geometry [70].

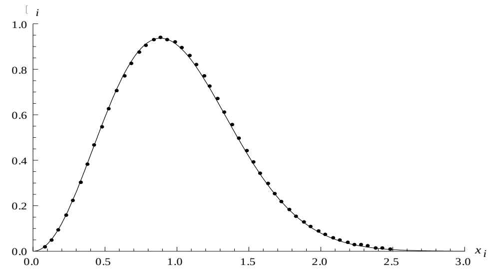


Fig. 4. Probability density of normalized spacings between zeroes $\gamma_n$ of Riemann zeta function $10^{12} \le n \le 10^{12} + 10^5$ [65], normalized spacings between eigenvalues of GUE [65], and the NMB distribution in Eq. (7).

Although the exact connection between noncommutative geometry and Riemann hypothesis is yet to be understood according to Connes [70]

*"The process of verification can be very painful: one's terribly afraid of being wrong…it involves the most anxiety, for one never knows if one's intuition is right- a bit as in dreams, where intuition very often proves mistaken"*

the ideas suggested above and further described in [35] may help in the construction of the physical foundation of such a mathematical theory.

## 6 Quantum Nature of Space, Time, and Dimension

The application of scale invariant model of statistical mechanics to the problems of infinitesimals and nonstandard analysis [72-75] resulted in the introduction of logarithmic coordinates and definition of "dimensionless" or "measureless" numbers as [76]

$$x_\beta = x'_\beta / \lambda_\beta \tag{8}$$

The "measure" $\lambda_\beta = \sqrt{\pi}/2$ was chosen [76] due to Gauss's error function on account of the equilibrium, i.e. random, distribution of particles (Fig. 1). According to Eq. (8) the range $(-1_\beta, 1_\beta)$ of the outer coordinate $x_\beta$ will correspond to the range $(-\infty_{\beta-1}, \infty_{\beta-1})$ of the inner coordinate $x_{\beta-1}$ leading to the coordinate hierarchy schematically shown in Fig. 5.

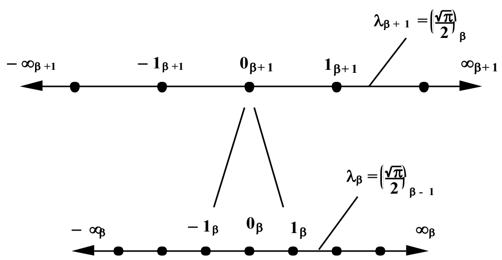


Fig. 5. Hierarchy of normalized coordinates for cascades of embedded statistical fields [76].

Following the classical methods [77-78] for systems of embedded statistical fields (Fig. 1) the logarithmic coordinate (Fig. 5) resulted in definition of invariant fractal dimension as [76]

$$D_{\beta+1} = N_{ES\beta-1} = N_{AE\beta} = \frac{\ln N_{AS\beta-1}}{\ln N_{AE\beta-1}} = -\frac{\ln N_{AS\beta-1}}{\ln r_{\beta-1}} \tag{9}$$

where $N_{ES\beta-1}$ and $N_{AS\beta-1}$ are the number of elements and atoms in the system, $N_{AE\beta-1}$ is the number of atoms in element, and $r_{\beta-1}$ is the size of the coarse–graining. Hence, dimension of physical space is identified as the number of embedded elements within its atom. Each (atom)$_{\beta+1}$ decompactifies into $N_{ES\beta-1}$ independent *(elements)$_\beta$* that constitute $D_{\beta+1}$ *dimensions* along which (atoms)$_{\beta-1}$ i.e. numbers could be placed similar to Cartesian coordinates (x, y, z). One notes that such "space" dimension $D_{\beta+1}$ will be energy dependent and since number of elements in each "atom" could be very large, fractal dimension of typical statistical field shown in Fig. 1 could be very large $10^7$ [79].

Invariant definitions for (system, element, atom) lengths $(L_\beta, \lambda_\beta, \ell_\beta)$ and velocities $(\mathbf{w}_\beta, \mathbf{v}_\beta, \mathbf{u}_\beta)$ lead to invariant definitions of (system, element, atom) "time" $(\Theta_\beta, \tau_\beta, t_\beta) = (L_\beta / w_\beta, \lambda_\beta / v_\beta, \ell_\beta / u_\beta)$ [36]. The instant, "now", or atomic time $t_\beta = 0_\beta$ of scale β will have a *finite duration* $t_\beta = \tau_{\beta-1}$ on a clock at the lower scale β–1 [36] leading to the following hierarchies of *time elements* for the statistical fields shown in Fig. 1.

$$\ldots\ \tau_e > \tau_c > \tau_m > \tau_s > \tau_k > \ldots \tag{10}$$

Clearly, the most fundamental and universal physical time is the time associated with the tachyon fluctuations $t_k = \tau_t$ [76] of *Casimir* vacuum [37]. Recently [32] Kelvin absolute temperature scale [degrees K] was identified as a length scale [meter] and related to particle de Broglie wavelength $T_\beta = <\lambda_\beta^2>^{1/2}$. Also, statistical time durations or periods are related to frequency $<\nu_\beta^2>^{1/2}$. Since particle energy is $\varepsilon_\beta = m_\beta <\lambda_\beta^2 \nu_\beta^2> = m_\beta <v_\beta^2> = kT_\beta$ it is clear that the connections between *space* and *time* and hence relativistic effects are *causal* [36, 50-52] and governed by thermodynamics [32, 48, 80] in accordance with Poincaré-Lorentz *dynamic theory of relativity* described in Section 3.

## 7 Concluding Remarks

If the signature of a good physical theory is its harmony with prior existing theories and empirical observations, then the scale invariant statistical theory of fields presented herein has been successful in satisfactory descriptions of relativistic effects, physical foundations of quantum mechanics, distribution of spacing of zeroes of Riemann zeta function, and quantum nature of both space and time hence justifying further development of the theory in all these areas.

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