On-off intermittency in the Ising model with temperature randomly varying in time

A. Krawiecki

Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland
(E-mail: akraw@if.pw.edu.pl)

Abstract. Ferromagnetic Ising model is investigated by means of Monte Carlo simulations, with temperature randomly varying in time, which assumes randomly values above and below the critical temperature in the consecutive simulation steps. It is known that for mean-field coupling on-off intermittency and attractor bubbling can be then observed, characterized by the sequence of laminar phases, during which the magnetization is almost zero, and chaotic bursts, during which the system becomes abruptly ordered. At the intermittency threshold distribution of the values of magnetization obeys a power scaling law. Here, possibility of the occurrence of analogous phenomena is studied in the Ising model on d-dimensional square lattices and on small-world networks which are obtained from the square ones by random rewiring of edges (corresponding to non-zero exchange integrals) with probability p. For the models on square lattices ($p = 0$) intermittent sequences of laminar phases and bursts of magnetization are observed only for $d \geq 4$; also only for $d \geq 4$ the distributions of values of magnetization exhibit power-law tails. For the models on small-world networks ($p > 0$) such distributions occur for $d \geq 2$. Thus, time series with certain properties of on-off intermittency can be observed close to the phase transition point in the above-mentioned generic models of statistical physics.
Keywords: on-off intermittency, attractor bubbling, Ising model.

On-off intermittency (OOI) appears in chaotoc systems in which the observed signal forms a sequence of laminar phases, during which it is almost constant and close to zero (the ”off” phase) and chaotic bursts (”on” state) [1,2]. The system can stay in the laminar phase for a long time, after which the burst can appear, i.e., rapid departure from, and return to, the ”off” state. OOI occurs in systems which posses a chaotic attractor contained within an invariant manifold with dimension smaller than that of the phase space. As a control parameter is varied, this attractor can lose transverse stability as a result of a supercritical blowout bifurcation [3], and a new attractor is formed which encompasses that contained within the invariant manifold. Just above the bifurcation threshold the phase trajectory spends most of the time in the vicinity of the invariant manifold and only occassionally departs from it, which results in the sequence of the laminar phases and bursts. In turn, if during the laminar phases instead of approaching zero the signal shows fluctuations with

Received: 29 March 2012 / Accepted: 5 October 2012

ISSN 2241-0503

amplitude small in comparison with that of chaotic bursts, the corresponding phenomenon is called attractor bubbling (AB) [2,4]. AB appears in systems with OOI under the influence of the internal or external noise, which amplifies local transverse inastabilities in the attractor contained within the invariant manifold [2]. This results in the appearance of intermittent bursitng below the blowout bifurcation threshold. OOI and AB were observed in many nonlinear dynamical systems, e.g., in model maps with time-dependent control parameter [1], in systems of coupled chaotic oscillators close to the synchronization threshold, where the invariant manifold is the synchronization manifold [5], in chaotic dynamics of spin waves [6], microscopic models of financial markets [7,8], etc.

It is interesting to note that OOI and AB can occur in many-body systems of statistical physics, e.g., in the Ising and Ising-like models [8-10] or electroconvection of nematic liquid crystals [11], under the influence of random variation of external parameters (the temperature or the elecric voltage in the two above-mentioned cases, respectively). In particular the ferromagnetic Ising model with temperature randomly varying in time can switch intermittently between the paramagnetic and ordered phase, which results in the sequence of the laminar phases and bursts in the time series of magnetization, treated as the signal. In the latter case OOI and AB have been observed so far in the Ising model with mean-field (MF) coupling [9]. The purpose of this paper is to show that these phenomena can appear also if the MF approximation is not exact, e.g., in the d-dimesnional Ising model with $d = 2, 3, 4 \ldots$ and, possibly, a small fraction of random connections corresponding to long-range exchange interactions between distant spins. It should be emphasized that the Ising model is a stochastic one (Glauber thermal bath dynamics is used in the Monte Carlo (MC) simulations), and the intermittency typical of dynamical systems appears in it as a result of interactions among a large number of stochastic units (spins). Thus the name "emergent" OOI and AB can be given to this kind of intermittent phenomena.

The model investigated in this paper is the ferromagnetic Ising model on a network which can be either a usual d-dimensional square lattice, with $d \geq 2$, or a small-world network obtained from the d-dimensional square lattice by random cutting and rewiring of edges [12]. For the latter purpose, each edge of the square lattice is cut with probability p and rewired so that one (randomly selected) end remains attached to an old node while the other one is attached to a new node, chosen randomly from among all nodes in the network. Multiple connections between nodes, self-connections and cutting edges once rewired are forbidden. The probability p controls the degree of randomness in the network: in particular, for $p = 0$ the network is the d-dimensional square lattice, and for $p = 1$ it is a random graph. The spins σ_i, $i = 1, 2, \ldots N$, $N = L^d$, where L is the size of the original square lattice and N is the number of spins, have two possible orientations, $\sigma_i = \pm 1$, and are located in the nodes of the network. The exchange integral between the spins σ_i, σ_j is $J_{ij} = J > 0$ if there is an edge between nodes i, j, and $J_{ij} = 0$ otherwise; hence, for $p > 0$ a certain fraction of long-range interactions between spins is present. The Hamiltonian for the model is

$$H = -\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j. \tag{1}$$

MC simulations of the above-mentioned model are performed with temperature randomly varying in time, $T(t) = T_0 - T_1\xi(t)$, where the discrete time steps t are equivalent to the consecutive MC simulation steps (each step corresponding to asynchronous updating of all N spins), $\xi(t)$ is a random variable with uniform distribution on the interval $(0, 1)$, and T_0, T_1 are constants. The model obeys the Glauber thermal-bath dynamics, with the transition rates between two spin configurations which differ by a single flip of one spin, e.g., that in the node i, in the form

$$w_i(\sigma_i, t) = \frac{1}{2}\left[1 - \sigma_i \tanh\left(\frac{I_i}{T(t)}\right)\right], \tag{2}$$

where

$$I_i = J\sum_{j\in\mathcal{K}_i}\sigma_j \tag{3}$$

is a local field acting on the spin i, and the sum in Eq. (3) runs over all neighbors of the node i (in particular, in the case of the d-dimensional square lattice, corresponding to $p = 0$, there are $z = 4, 6, 8\ldots$ nearest neighbors for $d = 2, 3, 4\ldots$, where z is the coordination number). Let us emphasize that the transition rates (2) depend on time due to the time dependence of the temperature $T(t)$.

For $T_1 = 0$ the models under study with $d = 2, 3, 4\ldots$ show ferromagnetic phase transition for $p \geq 0$, and the critical temperature T_c for given d is an increasing function of p. The order parameter is, of course, the magnetization $M = N^{-1}\sum_{i=1}^{N}\sigma_i$. Henceforth in the MC simulations it is always assumed that $T_0 > T_c$ for given d, p and the network size N. For $T_1 > 0$ the magnetization cannot be treated as the (static) order parameter since it can become dependent on time, in particular if $T_0 - T_1 < T_c$. Instead, the statistical properties of the time series $M(t)$ can be analyzed to search for the occurrence of the OOI or AB.

In the MF approximation, and in the thermodynamic limit, the equation for the time dependence of the magnetization becomes

$$M(t+1) = \tanh\left(\frac{J\langle z\rangle}{T(t)}M(t)\right) \approx \frac{J\langle z\rangle}{T(t)}M(t), \tag{4}$$

where $\langle z\rangle$ is the average coordination number ($\langle z\rangle = z = 4, 6, 8\ldots$ for $p = 0$ and $d = 2, 3, 4\ldots$), and the approximate equality is valid for $M \approx 0$. For $T_1 = 0$ and $T(t) = T_0 = \text{const}$ the magmetization $M(t)$ for $t \to \infty$ converges to zero if $T_0 > T_c^{(mf)} = \langle z\rangle J$, i.e., above the MF critical temperature, which corresponds to the paramagnetic phase, and to a non-zero value if $T_0 < T_c^{(mf)}$, which corresponds to the ordered phase. For $T_1 > 0$ Eq. (4) can be treated as a one-dimensional map describing the evolution of the magnetization $M(t)$ in discrete time t. This map possesses an invariant manifold $M(t) \equiv 0$, corresponding to the paramegnetic phase, and the temperature $T(t)$ is a (random) variable describing the dynamics of the two-dimensional system $(M(t), T(t))$ within this manifold. The map (4) belongs to a general class of systems $x_{t+1} = f(x_t, \eta_t)$ which after linearization in the vicinity of the invariant manifold $x_t \equiv 0$ have a form of multiplicative noise, $x_{t+1} = \eta_t x_t$, where η_t is a random variable. As the strength of the noise η_t rises the manifold $x_t \equiv 0$ loses stability via supercritical blowout bifurcation and OOI in the time series of x_t is observed; in Eq. (4), since $T_0 > T_c^{(mf)}$, this happens as T_1 is increased. In fact, OOI was observed in Eq. (4) as well as in the time series of magnetization obtained from MC simulations of the Ising model with temperature randomly varying in time and with MF coupling [9], where Eq. (4) is strict for $N \to \infty$, as T_1 was increased above the threshold value for the blowout bifurcation. Besides, in the MC simulations AB was also observed, i.e., chaotic bursts of magnetization which occurred for $T_1 < T_0 - T_c$, below the intermittency threshold, due to thermal fluctuations (internal noise) which destabilize the invariant manifold (the paramagnetic state) in finite-size systems.

In the cases studied in this paper neither OOI nor AB occur in the two- and three-dimensional Ising model on square lattices (for $p = 0$ and $d = 2, 3$, $T_0 > T_c$ and $0 < T_1 < T_c$): the magnetization exhibits only small fluctuations around zero (Fig. 1(a,c)). However, addition of even a small fraction of rewired edges ($p > 0$) leads to the occurrence of chaotic bursts in the time series of $M(t)$ typical of AB for large enough T_1 in the models with $d = 2, 3$ (Fig. 1(b)). In contrast, in the four-dimensional Ising model bursts in the time series of $M(t)$ occur both for $p = 0$ (the square lattice, Fig. 1(d)), if T_0 is slightly above T_c and T_1 is large enough, and for $p > 0$ (the small-world network, Fig. 1(e,f)), in a much wider range of the parameters T_0, T_1.

A characteristic feature of OOI is the distribution of lengths τ of laminar phases at the intermittency threshold, $P(\tau) \propto \tau^{-3/2}$ [1]; in the case of AB, due to the presence of noise, long laminar phases are less probable to occur and the tail of the distribution becomes exponential [4]. In the models studied in this paper, even for $N \simeq 10^4$, the thermal fluctuations were too strong to observe the power scaling law, and even for short laminar phases the distribution $P(\tau)$ decreased exponentially. Another characteristic feature of AB is the distribution of the values of the measured signal which exhibits power-law tails [13,14]. In the two- and three-dimesional Ising model on square lattices (for $p = 0$ and $d = 2, 3$, $T_0 > T_c$ and $0 < T_1 < T_c$) the distributions $P(M)$ have exponential rather than power-law tails (Fig. 2(a,b)), which confirms that no AB occurs. In contrast, in the Ising model on small-world networks with $d = 2, 3$ and $p > 0$ the tails of the distributions of magnetization obey the power scaling law, $P(M) \propto M^{-\alpha}$, $\alpha > 0$, for a certain range of $T_0 > T_c$ and large enough T_1 (Fig. 2(a,b)). In the four-dimensional Ising model on the square lattice ($d = 4$, $p = 0$) for T_0 just above T_c and large enough T_1 the distribution $P(M)$ obeys the power scaling law on a narrow interval; otherwise, $P(M)$ has exponential tails (Fig. 2(c)). For $d = 4$ and the small-world networks with $p > 0$ the tails of the distribution of the magnetization obey a power scaling law $P(M) \propto M^{-\alpha}$ for a wide range of the parameters T_0, T_1 (Fig. 2(d)). These results confirm that the occurrence of bursts in the time series of magnetization shown in Fig.

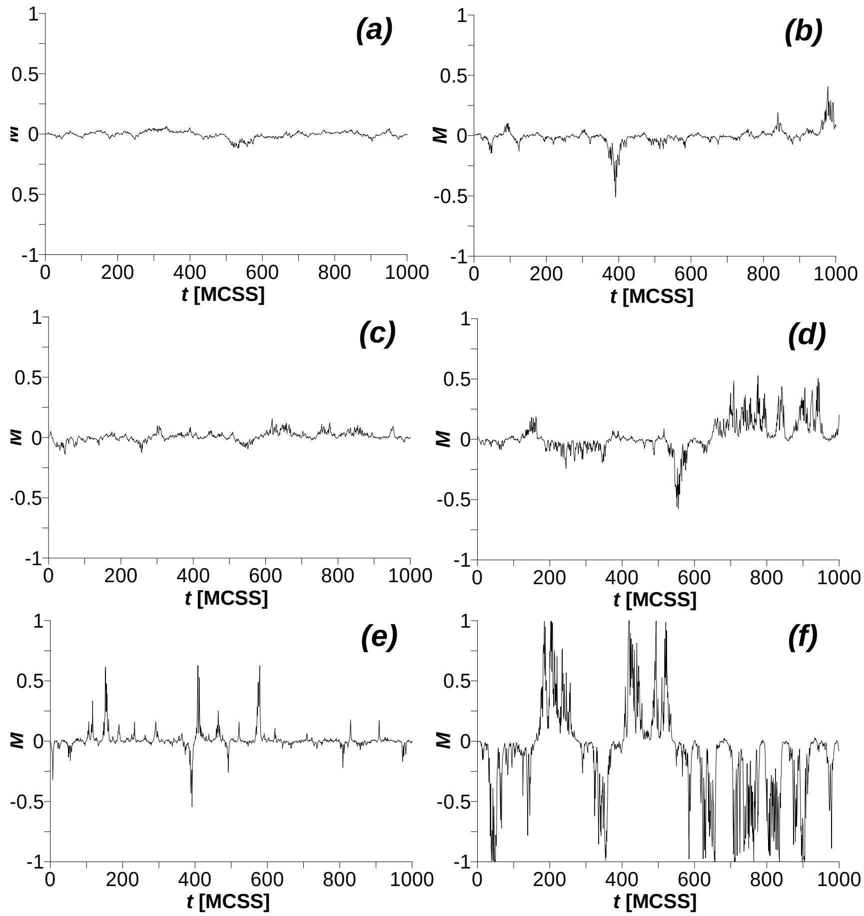

Fig. 1. Time series of magnetization $M(t)$ for the models with (a) $d = 2$, $L = 256$, $p = 0$ (two-dimensional square lattice), $T_0 = 2.45$, $T_1 = 2.44$; (b) $d = 2$, $L = 256$, $p = 0.2$ (small-world network obtained from the two-dimensional square lattice), $T_0 = 3.25$, $T_1 = 3.24$; (c) $d = 3$, $L = 40$, $p = 0$ (three-dimensional square lattice), $T_0 = 4.60$, $T_1 = 4.59$; (d) $d = 4$, $L = 16$, $p = 0$ (four-dimensional square lattice), $T_0 = 6.75$, $T_1 = 6.74$; (e) $d = 4$, $L = 16$, $p = 0.2$ (small-world network obtained from the four-dimensional square lattice), $T_0 = 10.0$, $T_1 = 9.99$; (f) $d = 4$, $L = 16$, $p = 0.2$, $T_0 = 7.35$, $T_1 = 7.34$.

1 (b,d,e,f) for the cases $d = 2, 3$, $p > 0$ and $d = 4$, $p \geq 0$ can be attributed to AB. In general, if the power scaling law $P(M) \propto M^{-\alpha}$ is observed the exponent $\alpha > 0$ decreases as T_0 approaches T_c from above and as T_1 is increased (Fig. 2(b,d)), since this leads to stronger and more frequent bursts in the time series of magnetization (Fig. 1(e,f)).

The above-mentioned results show that if the temperature varies randomly in time within a certain interval AB can be observed in the Ising model on small-world networks obtained from the two- and three-dimensional square

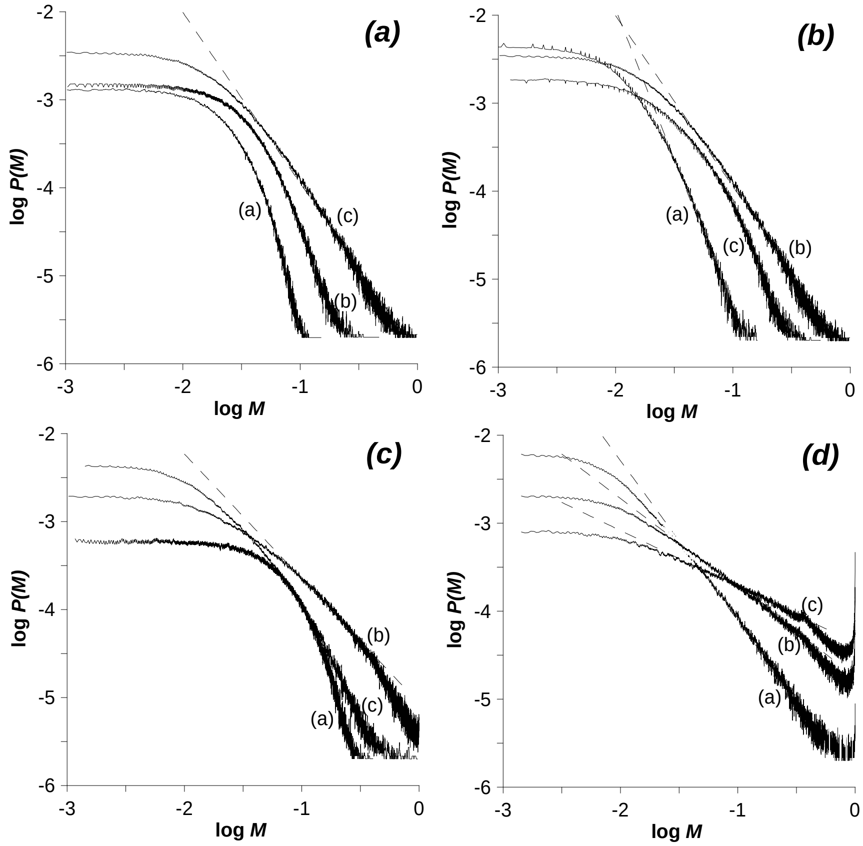

Fig. 2. Distributions of the magnetization $P(M)$ (solid lines) and possible fits of the power scaling laws to the tails of the distributions (dashed lines) for the models with (a) $d = 2$, $L = 256$ and $p = 0$, $T_0 = 2.45$, $T_1 = 2.44$ (curve (a)), $p = 0.2$, $T_0 = 3.25$, $T_1 = 1.00$ (curve (b)), $p = 0.2$, $T_0 = 3.25$, $T_1 = 3.24$ (curve (c)); (b) $d = 2$, $L = 256$ and $p = 0.2$, $T_0 = 4.25$, $T_1 = 4.24$ (curve (a)), $d = 2$, $L = 100$ and $p = 0.2$, $T_0 = 3.25$, $T_1 = 3.24$ (curve (b)), $d = 3$, $L = 40$ and $p = 0.0$, $T_0 = 4.60$, $T_1 = 4.59$ (curve (c)); (c) $d = 4$, $L = 16$, $p = 0$ and $T_0 = 6.75$, $T_1 = 1.0$ (curve (a)), $T_0 = 6.75$, $T_1 = 6.74$ (curve (b)), $T_0 = 7.75$, $T_1 = 7.74$ (curve (c)); (d) $d = 4$, $L = 16$, $p = 0.2$ and $T_0 = 10.0$, $T_1 = 9.99$ (curve (a)), $T_0 = 8.0$, $T_1 = 7.99$ (curve (b)), $T_0 = 7.35$, $T_1 = 7.34$ (curve (c)).

lattices by cutting and rewiring edges with probability $p > 0$. Due to the presence of shortcuts the interactions between spins have a MF character to some degree, but only for $p = 1$ the network becomes a random graph and the MF approximation, Eq. (4) becomes exact. Thus, AB can occur even if the MF approximation is not strict. In the four-dimensional Ising model AB can be observed both in the case of square lattice and the small-world network. Thus, the critical dimension for the occurrence of AB in the Ising model on a square lattice, with temperature randomly varuing in time, is $d = 4$.

It should be mentioned that a class of models similar to the ones considered in this paper was used in Ref. [10] to simulate the time series of price returns in the stock market. The two possible orientations of spins (agents) corresponded to the decisions to sell or to buy stocks, and instead of temperature varying randomly in time, exchange integrals between pairs of interacting agents varied randomly in time around the average which was also a random function of time. The agents were placed on a two-dimensional square lattice, and interactions with the first, second, third, etc. nearest neighbors were taken into account; then, small-world networks were also constructed by randomly cutting and rewiring edges with probability p. Parallel updating of the states of all agents was performed. Such Ising-like multi-agent models based on the social impact theory [15] are often used to reproduce so-called "stylized facts", or universal properites of the fluctuations of the stock prices [16]. In particular, the probability distributions of the stock price returns obtained from MC simulations, proportional to the magnetization, could exhibit power-law tails for $p > 0$, which is typical of the empirical distributions of returns. Also the time series of returns (magnetization) exhibited the empirically observed "volatility clustering", i.e., a sequence of quiescent (laminar) phases and bursts.

The results of the present paper as well as these of Refs. [8-11] confirm that "emergent" OOI and AB are ubiquitous phenomena in many-body systems of statistical physics. In the Ising model studied in this paper the appearance of OOI in the time series of magnetization can be easily understood within the MF approximation: as the amplitude of the stochastic variation of the external parameter (temperature) increases the invariant manifold $M = 0$, corresopnding to the paramagnetic phase, loses transverse stability as a result of the blowout bifurcation; in finite-size systems the occurrence of chaotic bursts of magnetization is facilitated due to interal noise (thermal fluctuations) and AB is observed. However, the results of the MC simulations show that similar intermittent phenomena occur even if the MF approximation is not exact.

References

1.N. Platt, E. A. Spiegel, and C. Tresser, *Phys. Rev. Lett.* 70:279, 1993; J. F. Heagy, N. Platt, and S. M. Hammel, *Phys. Rev. E* 49:1140, 1994; S. C. Venkataramani et al., *Physica D* 96:66, 1996.
2.P. Ashwin, J. Buescu, and I. N. Stewart, *Phys. Lett. A* 193:126, 1994.
3.E. Ott and J. Sommerer, *Phys. Lett. A* 188:39, 1994.
4.N. Platt, S. M. Hammel, and J. F. Heagy, *Phys. Rev. Lett.* 72:3498, 1994; S. C. Venkataramani et al., *Phys. Rev. Lett.* 77:5361, 1996.
5.A. Čenys et al., *Phys. Lett. A* 213:259, 1996.
6.F. Rödelsperger, A. Čenys, and H. Benner, *Phys. Rev. Lett.* 75:2594, 1995; A. Krawiecki and A. Sukiennicki, *Acta Phys. Polonica A* 88:269, 1995.
7.T. Lux and M. Marchesi, *Nature* 297:498, 1999;
8.A. Krawiecki, J. A. Holyst, and D. Helbing, *Phys. Rev. Lett.* 89:158701, 2002.
9.L. St epień and A. Krawiecki, *Phys. Stat. Solibi (b)* 236:511, 2003.
10.A. Krawiecki, *Int. J. Modern Phys. C* 19:1035, 2008.
11.T. John, R. Stannarius, and U. Behn, *Phys. Rev. Lett.* 83:749, 1999.
12.D. J. Watts and S. H. Strogatz, *Nature* (London) 393:440, 1998.

13.S. C. Venkataramani, T. M. Antonsen Jr., E. Ott, and J. C. Sommerer, *Phys. Lett. A* 207:173, 1995, *Physica D* 96:66, 1996.
14.P. Ashwin and E. Stone, *Phys. Rev. E* 56:1635, 1997.
15.B. Latané, *Am. Psychol.* 36:343, 1981; W. Weidlich, *Phys. Rep.* 204:1, 1991; D. Helbing, *Quantitative Sociodynamics* (Kluwer Academic, Dordrecht, 1995)
16.R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, UK, 1999); J.-P. Bouchaud and M. Potters, *Theory of Financial Risk* (Cambridge University Press, Cambridge, UK, 1999); H. Levy, M. Levy, and S. Solomon, *Microscopic simulations of financial markets* (Academic Press, New York, 2000); J. Voit, *The Statistical Mechanics of Financial Markets*, 2nd ed., (Springer, Heidelberg 2003).