Chaotic Modeling and Simulation (CMSIM) 4: 601-606, 2012

A Similar Nonlinear Telegraph Problem Governed By Lamé System

MEFLAH Mabrouk

Université Kasdi Merbah, Ouargla, Algérie
Laboratoire des Mathématiques Appliquées
Faculté des Sciences et de la technologie et
Des Sciences de la matière Ouargla 30.000 Algérie
mabrouk.meflah@univ-ouargla.dz

Abstract. Introducing the Lamé operator in the telegraph equation, we obtain theoretically a similar nonlinear system. In this work we are interested in the existence and uniqueness of function u=u(x,t), x ∈Ω , t ∈(0,T) solution for the new system by the elliptic regularization method.
Keywords: Lamé system, Elliptic regularization, Monotone operators,

1 Notations and position of the problem

Let Ω an open bounded domain of IR^n, with regular boundary Γ. We denote by Q the cylinder $IR_x^n \times IR_t$: Q = Ω ×]0,T[, with boundary Σ. L designed Lamé system define by μΔ + (λ+μ)∇div, λ and μ are constants Lamé with λ+ μ ≥0 and h,f are functions. We look for the existence and uniqueness of a function $u = u(x,t)$, $x \in \Omega$, t ∈]0,T[, solution of the problem (P)

$$
(P)\begin{cases} u'' + u' + u - Lu + |u'|^{p-2}u' = f \quad in\ Q & (1.1.1)\\ u = 0 \quad on\ \Sigma & (1.1.2)\\ u(x,0) = u(x,T) \quad \forall\ x\in\Omega & (1.1.3)\\ u'(x,0) = u'(x,T) \quad \forall\ x\in\Omega & (1.1.4)\end{cases} \qquad (1.1)
$$

2 Existence of the solution

Theorem1. Assume that Ω is bounded open of IR^n are given f, with f ∈ L(Q). Then there exists a function $u = w_0 + w$ satisfying (P)

$$w_0 \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \qquad (1.2)$$

$$w \in L^2(0,T;H_0^1(\Omega)) \qquad (1.3)$$

$$w' \in L^p(Q) \qquad (1.4)$$

Received: 3 April 2012 / Accepted: 12 October 2012

ISSN 2241-0503

Proof we use an approach due to G. Prodi [11] we have :

$$\begin{cases} u = w_0 + w \\ w_0 \text{ independent of } t \\ \int_0^T w\,dt = 0 \end{cases} \tag{1.5}$$

We introduce the Prodi idea (1. 5) in (1.1.1) we having

$$u'' + u' + u - Lu + |u'|^{p-2}u' - f = f + Lu_0 \tag{1.6}$$

We consider the derivative of (1.6) we obtain

$$\frac{d}{dt}(u'' + u' + u - Lu + |u'|^{p-2}u') = \frac{df}{dt} \tag{1.7}$$

And

$$\begin{cases} \int_0^T u\,dt = 0 \\ u(T) = u(0) \\ u'(x,0) = u'(x,T) \end{cases} \tag{1.8}$$

We deduce to (1.7)

$u'' - Lu + |u'|^{p-2}u' - f = h_0$ with h_0 independent of t

(1.9) For resolve (1.7) and (1.8) we denotes. L = -A; $\beta(u') = |u'|^{p-2}u'$

And we define the functional space V:

$$V = \begin{cases} v: v\in L^2(0,T,H_0^1(\Omega)); \quad v'\in L^2(0,T,(H_0^1(\Omega)) \cap L^p(Q); \\ v''\in L^2(0,T,L^2(\Omega)); \ \int_0^T v(t)dt = 0; \ v(T) = v(0); \ v'(T) = v'(0) \end{cases} \tag{1.10}$$

The Banach structure of V is defined by

$$\|v\|_V = \|v\|_{L^2(0,T,H_0^1(\Omega))} + \|v'\|_{L^2(0,T,H_0^1(\Omega))} + \|v\|_{L^p(Q)} + \|v\|_{L^2(0,T,L^2(\Omega))}$$

We define the bilinear form:

$$b(u,v) = \int_0^T [(u'',v) + (Au,v) + (\beta(u'),v)]dt \tag{1.11}$$

The weak formulation of (1.7) and (1.8) is to find $u\epsilon V$ such that

$$b(u,v) = \int_0^T (f,v')dt \quad \forall\ v\epsilon V \tag{1.12}$$

But (1.12) not coercive.

Then we following some ideas of Lions for obtain the elliptic regularization, given $\delta > 0$ and $u, v\ \epsilon V$, we define

$$\pi_\delta(u,v) = \delta \int_0^T [(u'',v'') + (Au',v')]ds + \int_0^T (u'' + Au + \beta(u'),v')ds. \tag{1.13}$$

The application $v \to \pi_\delta(u,v)$ is continuous on V so there exists an application $B_\delta\ \epsilon\ V'$: $\pi_\delta(u,v) = (B_\delta(u),v)$ (1.14)

The linear operator $B_\delta : V \to V'$ satisfies the four properties:

B_δ is bounded in V' for all bounded set in V and is a hemi continuous and is a strictly monotonous and is coercive.

In view of these properties and as consequence of theorem of Lions [4], there exist unique a function $u_\delta \epsilon\, V$:

$$\pi_\delta(u_\delta, v) = \int_0^T (f, v')dt \quad \forall\, v \epsilon V \tag{1.15}$$

2.1 A priori estimates I

Explicitly the elliptic regularization (1.15) and setting $v = u_\delta$, we obtain:

$$\delta \int_0^T [|u''_\delta|^2 + \|u'_\delta\|^2]dt + \int_0^T [|u'_\delta|^2 + (\beta(u'_\delta), u'_\delta)]dt = \int_0^T (f, u_\delta)dt \tag{1.16}$$

Or

$$\int_0^T (\beta(u'), u')dt = \|u'\|^p_{L^p(Q)} \; And \quad \int_0^T udt = 0 \;\Rightarrow \|u\|_{L^2\left(0,T,H_0^1\,(\Omega)\right)} \leq \mathrm{C}\, \|u'\|_{L^2\left(0,T,H_0^1\,(\Omega)\right)}$$

Then

u'_δ is bounded in $L^p(Q)$ when $\delta \to 0$ (1.17)

$$\delta \int_0^T \left[\left|u''_\delta\right|^2 + \left|u'_\delta\right|^2 + \left\|u'_\delta\right\|^2\right] dt \leq \mathrm{C} \tag{1.18}$$

Or

$\int_0^T u_\delta dt = 0.$ We have by (1.17) and (1.18) that:

u_δ is bounded in $L^p(Q)$ (1.19)

And

$$\delta \int_0^T \|u_\delta\|^2 dt \leq \mathrm{C} \tag{1.20}$$

2.2 A priori estimates II

Exchange in (1.15) v with:

$$v(t) = \int_0^T u_\delta(s)d - \frac{1}{T}\int_0^T (T - s)u_\delta(s)ds \tag{1.21}$$

We verify that:

$$\begin{cases} \int_0^T vdt = 0 \quad \forall v \in V \\ \quad v' = u_\delta \end{cases} \tag{1.22}$$

Taking into account (1.21) in (1.15) we get

$$\delta \int_0^T [(u''_\delta, u'_\delta) + (u'_\delta, u_\delta) + (Au'_\delta, u_\delta)]dt + \int_0^T [(u''_\delta, u_\delta) + (u'_\delta, u_\delta)]dt$$

$$+\int_0^T +\|u_\delta\|^2 dt + \int_0^T (\beta(u'_\delta), u'_\delta)dt = \int_0^T (f, u_\delta)dt \quad (1.23)$$

By using periodicity of $u_\delta, u'_\delta \epsilon V$, we obtain:

$$\int_0^T (u''_\delta, u'_\delta)dt = \int_0^T (Au'_\delta, u_\delta)dt = 0 \quad (1.24)$$

And

$$\int_0^T (u''_\delta, u_\delta)dt = (u'_\delta(T), u_\delta(T)) - (u'_\delta(0), u_\delta(0)) - \int_0^T (u'_\delta, u'_\delta)dt$$

$$= -\int_0^T |u'_\delta|^2 dt \quad (1.25)$$

By (1.24) and (1.17) we have

$$\left| \int_0^T (u''_\delta, u_\delta)dt \right| \leq \mathrm{C} \quad when \quad \delta \to 0 \quad (1.26)$$

Also, from (1.17) and (1.19) we obtain:

$$\left| \int_0^T (\beta(u'_\delta), u_\delta)dt \right| \leq \|\beta(u'_\delta)\|_{L^p(Q)} \|u_\delta\|_{L^p(Q)} \leq \mathrm{C}' \quad (1.27)$$

Combining (1.24), (1.26), (1.27) with (1.23) we deduce

$$\int_0^T \|u_\delta\|^2 dt \leq \mathrm{C} \quad (1.28)$$

2.3 Passage to the limit

From (1.17) and (1.28) that there exists a subsequence from (u_δ), such that

$$u_\delta \to 0 \quad weak\ in \quad L^2(0, T, H_0^1\ (\Omega)) \quad (1.29)$$

$$u'_\delta \to u' \quad weak\ in \quad L^p(Q) \quad (1.30)$$

$$\beta(u'_\delta) \to \chi \ weak\ in \quad L^q(Q) \quad (1.31)$$

Passage to the limit in (1.15) we obtain

$$\int_0^T [(-u', v'') + (Au, v') + (\chi, v')]dt = \int_0^T (f, v')dt \quad \forall v \in V \quad (1.32)$$

Use the convolution technical in (1.32) we have

$$\int_0^T (\chi, u' * \eta_\delta * \eta_\delta)dt = \int_0^T (f, u' * \eta_\delta * \eta_\delta)dt \quad \forall v \in V \tag{1.33}$$

When

$$\int_0^T (\chi, u'') dt = \int_0^T (f, u')dt \qquad \forall v \in V \tag{1.34}$$

3 Uniqueness of solution:

Theorem

Under the hypotheses of the theorem of existence, we consider two solutions u_1 and u_2 of the problem (P) then $u_1 = u_2$.

Proof: We subtract the equations (1.9) corresponding to u_1 and u_2 and sitting $\phi = u_1 - u_2$ we have:

$$\phi'' + A\phi + \beta(u_1') - \beta(u_2') \tag{2.1}$$

Denoting by (η_δ) the regularizing sequence a **similar** argument by Brézis [2] we obtain

$$\phi' * \eta_\delta * \eta_\delta = \phi * \eta'_\delta * \eta_\delta \tag{2.2}$$

Hence, by using (1.2) and (1.3), we have

$$\phi = \varphi + \phi_0 : \phi_0 \in V \ and \ \varphi \in L^2(0, T, H_0^1(\Omega)) \tag{2.3}$$

From (2.2) we get

$$\phi' * \eta_\delta * \eta_\delta = \phi * \eta'_\delta * \eta_\delta = \varphi' * \eta_\delta * \eta_\delta \tag{2.4}$$

Show that

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt = 0 \quad .$$

When

$$\int_0^T \frac{d}{dt}(\phi', \phi' * \eta_\delta * \eta_\delta)dt$$

$$= \int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt + \int_0^T (\phi', \phi'' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.5}$$

Therefore

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta)dt = \frac{1}{2}\int_0^T \frac{d}{dt}(\phi', \phi' * \eta_\delta * \eta_\delta)dt = 0 \tag{2.6}$$

ϕ' and η_δ periodic then we have

$$\int_0^T (\phi, \phi' * \eta_\delta * \eta_\delta)dt = \int_0^T (\phi', \phi' * \eta_\delta * \eta_\delta)dt = \int_0^T (A\phi, \phi' * \eta_\delta * \eta_\delta)dt \tag{2.7}$$

From (2.1); (2.6) and (2.7) we obtain:

$$\int_0^T (\beta(u'_1) - \beta(u'_2), \phi' * \eta_\delta * \eta_\delta) = 0 \tag{2.8}$$

Passage to the limit in (2.8) we have

$$\int_0^T (\beta(u'_1) - \beta(u'_2), u'_1 - u'_2)dt = 0 \tag{2.9}$$

Where

$$u'_1 - u'_2 = 0 \Rightarrow u'_1 = u'_2 \tag{2.10}$$

This implies that

$$\phi = u_1\text{-}u_1 = \theta,\ \theta \text{ independent of t} \tag{2.11}$$

From (2.7) *and* (2.11) *we obtain*

$$\int_0^T (A\theta, \theta)dt = 0 \quad \forall\theta \in V \tag{2.12}$$

We deduce from (1.2)

$$\theta \in H_0^1\ (\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \tag{2.13}$$

By (2.12) and (2.13) and using theorem of Green we have $(A\theta,\theta) = 0 \Rightarrow \theta = 0$. Where the uniqueness of solution.

References

1. R.A. Adams, Sobolev Spaces, Academic Press, (1976).
2. H. Brezis, Analyse fonctionnelle, théorie et applications. Masson (1983).
3. G.Duvaut, J.L. Lions, Les inéquations en mécanique et en physique. Dunod. Paris. (1972).
4. J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod. (1969).
5. Luc Tartar, Topics in non lineair analysis. Université de Paris-Sud, Publications Mathematiques d'Orsay, novembre (1978).
6. M. Meflah, Study of Nonlinear Elasticity Problem by Elliptic Regularization with Lamé System, Int. J. of Mathematical Archive-2(5), May 2011, Page 693-697, ISSN 2229-5046
7. M. Meflah, A Nonlinear Elasticity Problem by Elliptic Regularization Technics, Int. J. Contemp. Math. Sciences, Vol 6, 2011,no. 25, 1221-1229
8. M. Meflah, B Merouani, A Nonlinear Elasticity Problem Governed by Lamé System, Applied Mathematical Sciences, Vol.4, 2010, no. 36,1785-1796.
9. B. Merouani, M. Meflah, A. Boulaouad, The Generalized and perturbed Lamé system, Applied Mathematical SciencesVol.2, 2008,no. 49,2425-2430
10. F. Messelmi, B. Merouani, M. Meflah, Nonlinear thermoelasticity problem, Analele Universitatii Oradea, Fasc. Matematica,Tom XV (2008), 207-217.
11. G. Prodi, Soluzioni periodiche dell'equazione delle onde com termine dissipativo nonlineare.Rend. Sem.Mat. Pandova, 35 (1965).
12. V. Patron, P. Perline, Méthode de la théorie mathématique de l'élasticité, Editions Mir, Moscou, 1981.
13. Taylor M.E, Partial differential non linear equations III Springer. (1996).