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Nonlinear Dynamical Systems Theory and Economic Complexity

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Abstract: Catastrophe theory and deterministic chaos constitute basic elements of the science of complexity. Elementary catastrophes were the first form of nonlinear, topological complexity that were seriously studied in economics. Deterministic chaos and other types of complexity succeeded catastrophe theory. In general, chaos means the seemingly random behavior of a deterministic system, which stems from high sensitivity to its initial conditions. Nonlinear dynamical systems theory, which unites various manifestations of complexity into one integrated system, is contrary to the assumptions that markets and economies spontaneously strive for a state of equilibrium. To the contrary, their complexity seems to grow due to the influence of classic economic laws. In my paper, I indicate that with time, model economic systems strive for a state we call "the edge of chaos". I consider two cases. The first case concerns an economy based on a two-stage accelerator, where the economic cycle adopts the form of chaotic hysteresis. The second case concerns a Cournot-Puu duopoly model in which striving for the edge of chaos stems from profit maximization by entrepreneurs. The evolution of systems at the edge of chaos can be sudden, which makes it necessary to consider it in terms of elementary catastrophes.
Keywords: Cusp catastrophe, Chaotic hysteresis model, Cournot-Puu duopoly model, Edge of chaos, System classification, Economic transformation, Rule of progressive complexity.

1. Introduction: Foundations of catastrophe theory

1.1. Classification Theorem

The theory of catastrophes, also known as the theory of morphogenesis, appeared in science in the mid-1970s [25]. It is a general method of system modeling focusing on the way in which discontinuous effects can emerge from continuous causes. Let the dynamic system be represented by a smooth function:

$$f : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}, \tag{1}$$

where $\mathbf{R}^k$ is a control space representing a set of causes, whereas $\mathbf{R}^n$ is a space of states (behavior) representing a set of effects. The function *f* is called a potential function. If the internal dynamics of the system consist in striving for a

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local maximum, then the potential function can represent the probability of it being found.
The basis of catastrophe theory is the classification theorem [26]. This states that if the co-dimension k of elementary catastrophes is bigger than 5, they create a finite family of discontinuous transition types. Every sudden dynamic change can be assigned to one of those types. The relation between the number of catastrophes and the co-dimension is shown in Table 1.

Table 1. Elementary catastrophe classification in relation to the co-dimension k

Co-dimension value (k)	1	2	3	4	5	6	7 …
Number of elementary catastrophes	1	2	5	7	11	∞	∞

From an application point of view, the case $k = 4$ is important, since $\mathbf{R}^4$ can be interpreted as a physical space-time in which all events take place. There are seven types of singularities in this case: fold, cusp, swallowtail, butterfly, hyperbolic umbilic, elliptic umbilic, and parabolic umbilic [5].
The application of the catastrophe theory in economics is possible only when the law governing a given phenomenon or process has been well-defined. In such a case, the catastrophe theory will facilitate the choice of the easiest mathematical structure, which will generate a behavior closest to real. Another equally point is to use metaphors properly [8].

1.2. The cusp catastrophe
The cusp catastrophe is one of the most common elementary catastrophes in economic applications. The potential function has the following form:

$$f : \mathbf{R}^2 \times \mathbf{R}^1 \to \mathbf{R}, \tag{2}$$

Thus, the control space is two-dimensional, whereas the state space is one-dimensional. The function (2) has a simple multinomial representation:

$$f(c_1, c_2, x) = \frac{1}{4}x^4 + \frac{1}{2}c_1 x^2 + c_2 x, \tag{3}$$

where x stands for the state variable, whereas c_1 and c_2 are the control parameters [28]. The manifold of the catastrophe defining the surface area of the system equilibrium is dependent on the following formula:

$$M_3 = \left[(c_1, c_2, x) : \frac{df}{dx} = 0, \frac{df}{dx} = x^3 + c_1 x + c_2 \right]. \tag{4}$$

The system proceeds along this surface in a continuous way, until it comes across a set of singularities. There is then a sudden jump to another equilibrium surface and the continuous evolution continues until the next jump.

2. Deterministic chaos

2.1. Nonlinearity as a necessary condition for complexity

In order to define nonlinearity it is necessary to clearly define linearity. In all linear systems, the binding rule is the rule of superposition. This states that the system's reaction to two or more stimuli is the sum of reactions triggered individually by each of these stimuli. If factor *A* triggers reaction *X*, and factor *B* reacts to *Y* then the factor (*A* + *B*) results in (*X* + *Y*). In other words, linear systems are additive.

The rule of superposition implies the linearity of the system if we supplement it with the condition of homogeneity. A lack of additiveness and homogeneity implies the nonlinearity of the system. The main causes of nonlinearity in economics are:

- Limitations imposed on the economic variables [2].
- Technical-balance laws of production [15].
- Technical-organizational factors [10].
- Bounded rationality [24].
- Processes of expectation formation [4].
- Adaptive processes of economic-agent learning [3].
- The shape (protuberance) of the indifference curves.
- Aggregation processes of some variables [27].
- Evolution of competition rules [3].
- Psychological laws [14].

Nonlinearity is a necessary condition, but it is not enough to trigger chaos. Statistical tests confirm that nonlinearity is a phenomenon that is common in economic time series, and part of them proves that deterministic chaos exists. There are strong grounds to claim that in the future, the role of nonlinearity in economic explorations will become more significant.

2.2. The butterfly effect

Deterministic chaos means a seemingly random behavior of the deterministic system, thus one which is strictly subject to specific rules. The reason for the stochastic behavior of some nonlinear deterministic systems is their unusually sensitive dependence on initial conditions, which was named 'the butterfly effect' by Lorenz [16]. A slight disturbance in the initial conditions after some time causes significant changes in the system behavior as trajectories begin to disperse exponentially. As picturesquely described Lorenz, a proverbial flap of butterfly wings in Brazil can cause a tornado in Texas.

The Lyapunov exponents are amongst the most frequently-used quantitative measures of the trajectory divergence. This notion has been used by Oseledec [20] in a well-known multiplicative ergodic theorem. The Lyapunov exponent for one-dimensional map is as follows:

$$W^L = \lim_{n\to\infty} \lim_{\varepsilon\to 0} \frac{1}{n} \ln \left| \frac{f^n(x_0+\varepsilon) - f^n(x_0)}{\varepsilon} \right| = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{d\, f^n(x_0)}{d\, x_0} \right|. \quad (5)$$

Symbols $f^1, f^2, \ldots, f^n$ stand for subsequent iterations, x_0 and $x_0+\varepsilon$ are the initial conditions for the two trajectories. The number $\varepsilon > 0$ is very small. With every iteration, the distance between the trajectories increases exponentially. This definition can also be generalized with multi-dimensional systems. The number of exponents has to correspond to the number of dimensions. If the largest exponent of a dynamical system is positive, this indicates a chaotic trajectory, while an exponent equal to zero indicates the bifurcation point, and a negative value means convergence of the trajectory with the constant point of attraction or a periodic attractor.

The basic notion of nonlinear dynamical systems theory is also the notion of an attractor, primarily a chaotic attractor. Let *F* stand for a map of *m*-dimensional space into itself. The compact set *A*, which is situated in the *m*-dimensional space, we call the attractor for *F* if it meets the conditions of invariance, density, stability and attraction. An attractor is a chaotic attractor if it contains a chaotic trajectory [19].

2.3. System classification in nonlinear dynamical systems theory

In order to compare the subjects of conventional science, the theory of deterministic chaos, and the theory of complexity, we can classify systems based on the following criteria: the number of constituents of the system *N* and the average number of links between these elements *K* (see [11, 12, 13]). Depending on the relationship between these parameters, we can distinguish three types of the *NK* systems:

- Type I – subcritical systems. The number of links is very small, given the number of elements. Every element is technically independent from others, thus the behavior of the whole system can be treated as a simple sum of its parts. Because the rule of superposition is met in such a case, systems of this type are approximately linear. Their dominating behavior is striving for states of equilibrium.
- Type II – critical systems. The average number of links is substantially greater than in the subcritical systems. These systems are characterized by more complex dynamics and can reveal emergent properties [7]. Local changes can be dispersed in a system so they usually do not bring about global consequences. These types of systems often balance on the edge of chaos (this is a state when the system’s ability to survive is the greatest and its computing power reaches maximum value).
- Type III – supercritical systems. The ratio of the number of links to the number of elements approaches one. It is a state in which almost every

element is interlinked with all the rest. It includes deterministic systems, which are characterized by complex dynamics.

The largest Lyapunov exponent for subcritical systems is negative, for critical systems it oscillates around zero, whereas for supercritical systems it is positive. Classical science deals with systems of type I, the theory of chaos explores systems of type III, whereas the subject of interest for the theory of complexity is type II and the transitions between various types of systems (see [7, 21]).

3. Applications in economics

3.1. The theory of economic transformation

The first step towards elaborating a theory of transformation was taken by American researchers who formulated a model of chaotic hysteresis (see [22, 23]). Two basic nonlinear dynamical systems theory methods were applied concurrently, i.e. elementary catastrophes and deterministic chaos. The starting point is a socialist economy. According to the Marxist convention, the economy was divided into two sectors: consumption-goods and capital-goods. The notion of a technological gap and the cusp catastrophe were used to describe social-economic crises. The attractor in the form of a chaotic hysteresis that appears in a reformed economy is a result of a two-phase activity of a nonlinear accelerator.

The dynamic system is described by a two-dimensional formula:

$$I_t = I_{t-1} + Z_t, \tag{6}$$

$$Z_t = u\left(Z_{t-1} - Z_{t-1}^3\right) - v I_{t-1}, \tag{7}$$

where: I_t – total investment within the period *t*, Z_t – increase in the investment, whereas symbols *u* and *v* means respectively the values of accelerators in the capital-goods sector and in the consumption-goods sector. These formulas cannot be solved analytically, but they can be the subject of numerical explorations.

An analysis of the system (6)–(7) was conducted assuming the constant value of the accelerator in the sector of capital goods $u = 2$, whereas the value of the parameter *v* was gradually decreased. For $0.01 \le v \le 0.1395$ in the phase space of the system there is an investment cycle in the form of a chaotic attractor. Lowering the value of the accelerator of the consumption-goods sector means the metamorphosis of the attractor – eventually for the value of $v = 0.00005$ it takes on the form of chaotic hysteresis. The attractor in this form is featured in Figure 1. In the model, there is a trade-off between complexity (chaos) and instability, understood as the increase of period and amplitude of oscillation of investment [6].

The next element of the theory is the technological gap (*G*), which stems from the higher rate of capital-intensive nature of production in socialism compared to a capitalist economy. Paradoxically, this phenomenon is a result of pursuing the postulates of stability of production and full employment, which were to make socialism a system more bearable for people than capitalism with its chronic unemployment and crises.

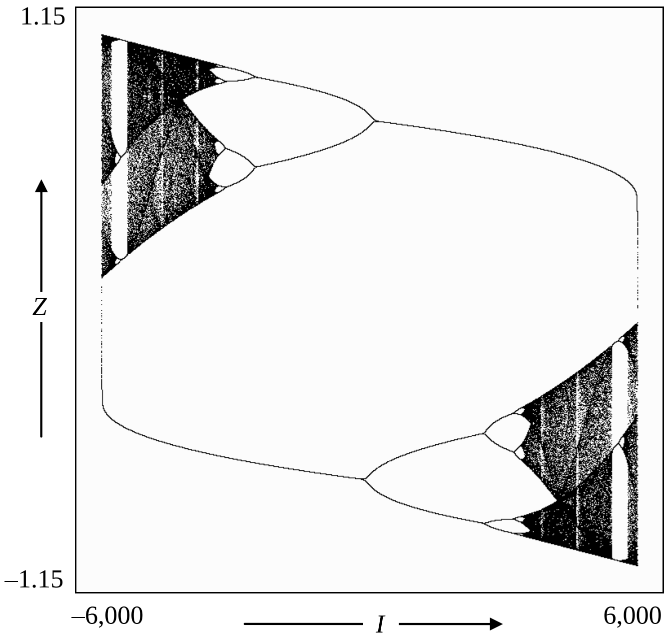

Figure 1. The chaotic attractor in the form of chaotic hysteresis for $u = 2$, $v = 0.00005$

Another step is to introduce the cusp catastrophe, whose space area of equilibrium meets condition (4). In the theory under investigation, the variable of this state is the probability of an introduction of market reforms $x = P(s)$, the bifurcation parameter is the dimensions of the technological gap $c_1 = G$, whereas the asymmetric parameter is the rate of growth of investment $c_2 = Z/I$. In the Figure 2 there is a geometrical interpretation of the morphogenetic model of transformation.

The space of the catastrophe equilibrium describes various scenarios of economic crises and the corresponding reforms that sought to answer them. For $G = 0$ we have an example market economy. The occurrence of the technological gap, which happens after passing through the beginning of catastrophe, causes a division of the equilibrium space into two layers – an upper and lower. They suggest the occurrence of nonlinear changes in the probability of transformation, whenever the rate of investment growth reaches a necessary value. Sudden leaps take place when the asymmetric factor crosses the bifurcation set of the catastrophe located in the parameter space marked by the following formula:

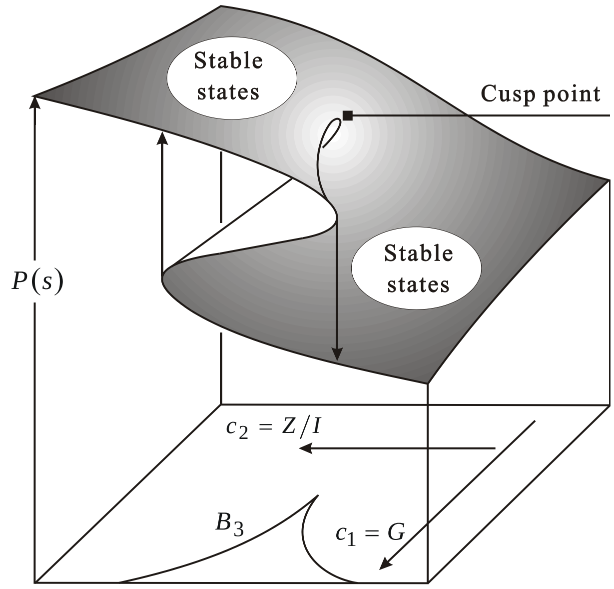

Figure 2. Geometrical interpretation of a morphogenetic transformation model

$$B_3 = \left[(c_1, c_2) : 4c_1^3 + 27c_2^2 = 0\right]. \tag{8}$$

Numerical explorations of model (6)–(7) have shed new light on a certain macroeconomic problem which has been neglected by mainstream economics regarding the macroeconomic costs of the reform complexity. An intuitive understanding of this category of costs is known from the theory of the corporation [18]. The global financial crisis prompted a wider look at the complexity of economic processes and the accompanying problems [1]. An economy under transformation is vulnerable to falling victim to trade-offs between complexity and instability, which accounts for the fact that benefits stemming from reforms can, over a long period of time, consolidate below the costs of complexity. It is a new, quality-based position in the balance of transformation. Future research should focus on methods of its measurement. In addition, it constitutes a challenge to economic policy, which should seek to simplify economic life.

3.2. The rule of progressive complexity

Mathematical studies of standard nonlinear economic models have revealed an interesting regularity, which I called “the rule of progressive complexity” [9]. It appears that there are two active forces in economic systems. The first force is short term in nature, and its source stems from rational, typical endeavors of business entities. One of the manifestations of this activity is profit maximization by producers and maximization of utility by consumers. As a result, these systems seek a state of short-term equilibrium. The second force is active over a long period of time and even though its source is identical to the first one, the effects are totally different. It destabilizes the short-term states of equilibrium and pushes market structures towards a state known as "the edge of chaos". It is a transition field between a periodic behavior and chaotic behavior, where the computing power of systems, which means their ability to collect and process information, reaches its maximum. The complexity of a system, which can been measured by Lapunov exponents, increases in this field.

Let us consider a duopoly model using the following equations:

$$x_{t+1} = \sqrt{\frac{y_t}{a}} - y_t, \tag{9}$$

$$y_{t+1} = \sqrt{\frac{x_t}{b}} - x_t, \tag{10}$$

where: x – the production output of the first entrepreneur, y – the production output of the second entrepreneur, whereas a and b stand for their marginal costs, respectively. In the static version, these equations set the reaction functions. Each of them describes the choice of the production output made by an entrepreneur assuming that the production output of their competitor is known. The collision of these two functions takes place at the point known as the Cournot-Nash equilibrium point.

The standard analysis of the model's stability allows us to set two critical values of the marginal costs ratio:

$$\frac{a}{b} \vee \frac{b}{a} = 3 \pm 2\sqrt{2}\,. \tag{11}$$

This is where the analytical methods give up. We do not know what happens to this model when the stability threshold is crossed, or how it behaves over a long period.

It is best to start numerical explorations of a duopoly (9)–(10) with making a period plot [19]. This is a two-dimensional space of parameters in which various behavior of the system has been specified (with emphasis on periodic behavior). In order to do this, one should define the interval of changeability of both parameters and the initial condition of the trajectory bundle. A plot of this type allows us to follow the dynamics of the system depending on a simultaneous change in two control parameters.

Numerical explorations of parameter space reveal the following types of behavior: states of short-term equilibrium, periodic dynamics, chaos and divergent trajectories (see Figure 3). Pairs of parameters responsible for states of stable equilibrium account for 82.77% of the parameter space, whereas pairs of

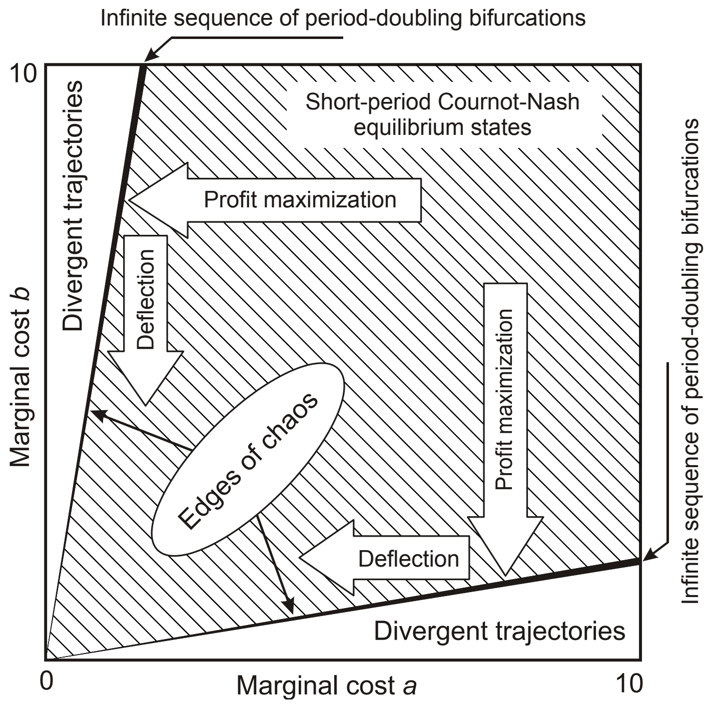

Fig. 3. Dynamics of the Cournot-Puu duopoly in the parameter space

chaotic parameters account for mere 0.15% of this space. Consequently, it seems that stability predominates and the claims of conventional economics have been confirmed. However, it is a false conclusion. Entrepreneurs are interested not only in maximizing profit over a short time, but also in the long run. Maximizing profit in the long run requires introducing technical-organizational progress and it results in lowering marginal costs. Consequently, every producer strives for one of the two edges of chaos (11), i.e. states with growing complexity [9].

The system displays a certain type of globally rational behavior which contributes to its survival. As of the moment the efficient producer achieves the edge of chaos, his long-term profit decreases, and the long-term profit of the inefficient producer begins to grow [17]. This leads to role reversal, and in the diagram, the market bounces off the edge of chaos.

4. Conclusions

Catastrophe means a violent, sudden transition of the tested system into a new state. What is important here is the rapidity of the changes in the behavior of an object as compared with the mean change in the past. Catastrophe theory merges two apparently contradictory and unrelated kinds of phenomena descriptions to form one coherent notion system: evolutionism and revolutionism, continuity and discontinuity. In economics, the application of catastrophe theory is of great cognitive importance, particularly in issues of explanation and forecasting in economics.

In transitional economies, there is a trade-off between complexity and instability. In the economic calculation of transformations, a new type of cost needs to be considered – the social costs connected with the change of the dynamic complexity of the systems. Numerical explorations of an archetypal duopoly model have proven that states of equilibrium are stable only for a short period. In the long run, such systems strive for the edge of chaos.

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