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Intermittency in the Generalized Lorenz Model

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Abstract. We consider a low-dimensional model of hydromagnetic convection in a horizontally magnetized layer of a viscous fluid heated from below, which is a generalization of the standard Lorenz model. We analyze the stability of the fluid influenced by the induced magnetic field. By changing two control parameters related to the temperature difference and the applied magnetic field strength, one can see various transitions from regular to irregular long-term behavior of the system through intermittency scenario. We discuss bifurcations leading to both type I and III intermittency. We therefore hope that this model could shed new light on dynamics of magnetohydrodynamic convection.
Keywords: Chaos, Intermittency, Convection, Magnetohydrodynamics, Turbulence.

1 Introduction

Dynamics of viscous fluids is still a challenging question in physics of fluids. For example, a simple case of the Rayleigh-Bénard convection is known for hundred years [6]. The important breakthrough happened fifty years ago, when starting from complex basic hydrodynamic equations Lorenz obtained three simple but nonlinear ordinary differential equations [7]. This seminal well-known paper has revealed complexity of nonperiodic chaotic deterministic flow, including strange attractors, bifurcations, and intermittency. Five years ago we have generalized this model for convection in a horizontally magnetized fluid layer by adding a new variable responsible for the magnetic field embedded in the fluid [8]. Solutions of this still simple four-dimensional nonlinear deterministic system can exhibit rather complex nonperiodic behavior, but surprisingly the influence of the applied magnetic field is not quite trivial. In particular, it appears that by changing control parameters the system can easily go from equilibrium (fixed point) or periodic to nonperiodic chaotic or even hyperchaotic behavior [9]. Naturally, besides of the transitions induced by the parameters changes, all these types of behavior can be intertwined due to intermittent character of dynamics.

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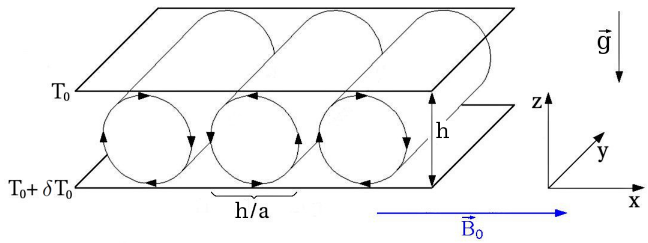

Fig. 1. Schematic of geometry for Rayleigh-Bénard convection cells

Within the theory of dynamical systems transitions from fixed points to periodic or nonperiodic flows often occur in a given system through intermittency scenario, where signals alternate between regular (laminar) phases and irregular bursts. Based on various characteristic of dynamical systems, three basic types of intermittency have been distinguished in the scientific literature. Namely, types I, II, and III are related to saddle-node, Hopf, and inverse period doubling bifurcations, correspondingly [13]. These basic types of intermittent behavior can be verified experimentally by looking at their statistical properties. More recently other intermittency mechanisms have also been found, including, e.g., on-off intermittency [12], eyelet intermittency [11], and ring intermittency [4]. In this paper we discuss in detail types I and III intermittency identified in the generalized Lorenz model.

2 The generalized Lorenz model

The schematics of the standard Rayleigh-Bénard cells in two-dimensions [6] in a horizontal (x axis) viscous fluid layer of height h and aspect ratio a is shown in Fig. 1 (no variations in y direction), cf. e.g., Appendices to Refs. [1,14]. The external gravitational field $\mathbf{f}$ equal to a constant acceleration $\mathbf{g}$ acting vertically (along z axis) on the fluid of mass density ρ results in the buoyancy term in the equation of motion, $f = \rho g$. The fluid is heated from below with an initially applied vertical (z axis) temperature gradient, δT_0. As usual, using a constant coefficient β, we take into account the volume expansion for f term, $\rho = \rho_0[1 - \beta(T - T_0)]$, but except that the fluid is treated as incompressible, $\rho = \rho_0$ (the Oberbeck-Boussinesq approximation) [10,3].

In the standard three-dimensional Lorenz model, besides a time-dependent variable X proportional to the intensity of the convective motion, the other two variables Y and Z describe the temperature profile, see Ref. [7]. In addition, in the case of the magnetized fluid we have introduced a new time dependent variable W describing the profile of the magnetic field induced in the convected fluid. One can expect that in the case of a thin horizontal layer, the influence of an external horizontal magnetic field should be important. Hence we apply an initial magnetic field $\mathbf{B_0}$ along the x direction.

In this case, by using a reasonable approximation, $(\mathbf{B} \cdot \nabla)\mathbf{v} \approx (\mathbf{B_0} \cdot \nabla)\mathbf{v}$ in magnetic advection equation, we have obtained from magnetohydrodynamics theory [5], described by partial differential equations consisting of Navier-Stockes equation of motion, magnetic advection-difusion equation, and heat conduction equation, a generalized model described by four ordinary differential equations [8]:

$$\dot{X} = -\sigma X + \sigma Y - \omega_0 W, \tag{1}$$

$$\dot{Y} = -XZ + rX - Y, \tag{2}$$

$$\dot{Z} = XY - bZ, \tag{3}$$

$$\dot{W} = \omega_0 X - \sigma_{\mathrm{m}} W, \tag{4}$$

where dots denote derivatives with respect to the normalized time $t' = (1 + a^2)\kappa(\pi/h)^2\, t$, while $\sigma = \nu/\kappa$ is the Prandtl number (ratio of the kinematic viscosity and the thermal conductivity), and $b = 4/(1+a^2)$ is the geometrical factor for a given fluid. As usual $r = R_{\mathrm{a}}/R_{\mathrm{c}}$ is a control parameter of the dynamical system proportional to the temperature gradient δT_0, or a Rayleigh number $R_{\mathrm{a}} = g\beta h^3 \delta T_0/(\nu\kappa)$ normalized by a critical number $R_{\mathrm{c}} = (1 + a^2)^3(\pi^2/a)^2$.

In addition to the standard Lorenz system [7], we have introduced another control parameter proportional to the initial magnetic field strength B_0 applied to the system, which is defined as a basic dimensionless magnetic frequency $\omega_0 = v_{\mathrm{A}0}/v_0$, with the Alfvén velocity $v_{\mathrm{A}0} = B_0/(\mu_0\rho_0)^{1/2}$ (using the constant magnetic permeability μ_0) and $v_0 = 4\pi\kappa/(abh)$ [8,9]. The last term in Eq. (1) comes from the anisotropic tension of the magnetic field $(\mathbf{B}\cdot\nabla)\mathbf{B}/(\mu_0\rho)$ in the equation of motion. Naturally, besides the Prandtl number $\sigma = \nu/\kappa$, the properties of the magnetized fluid are characterized by an analogue parameter $\sigma_{\mathrm{m}} = \eta/\kappa$, where η denotes magnetic diffusive viscosity (resistivity), appearing in Eq. (4) that results from the magnetic advection and diffusion terms in the respective magnetohydrodynamics equations.

3 Intermittency

It is worth noting that still in a chaotic regime but in the proximity of the boundary between chaotic and periodic region we have identified intermittent behavior of the system illustrated in Fig. 2, where almost periodic oscillations are interrupted by bursts of irregular behavior [8,9]. This phenomenon of intermittency can be observed as bursts of increased energy dissipation, defined here as $\nu|\mathbf{v}|^2 + \eta|\mathbf{B}|^2/(\mu_0\rho)$. By analysis of a Poincaré map (constructed from the values of Y variable taken for $X = 0$ plane crossings) we have identified this intermittency as type I and III, see Ref. [13]. The intermittency of these types displays characteristic behavior of the signal, distribution of lengths of laminar intervals, and dependence of the mean length of laminar interval on bifurcation parameter as described thoroughly, e.g., in Ref. [14]. In this context, analysis of statistical properties (e.g. distributions or scaling in intermittency) of the observed dynamical behavior can be more interesting from experimental point of view; thus we discuss the statistics below.

Next, we determine the lengths of laminar phases and their distribution using an algorithm, where pieces of a long numerical solution are compared to a periodic (laminar) phase pattern in four-dimensional phase space. The

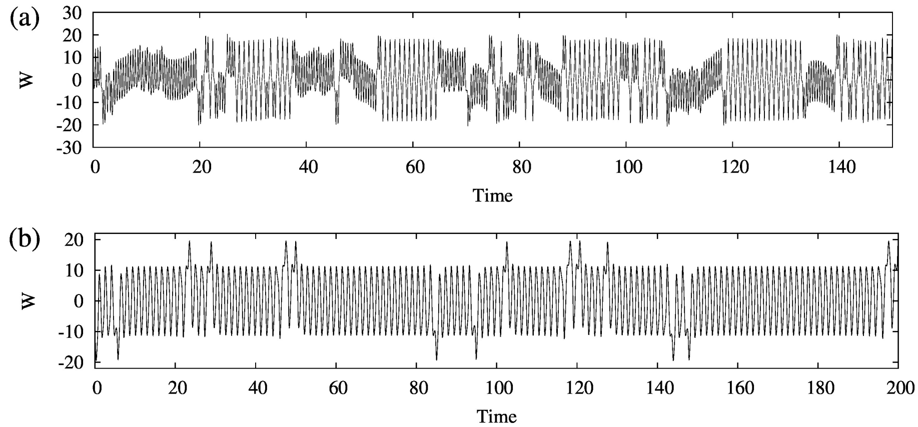

Fig. 2. The intermittent behavior of the variable W for the generalized Lorenz model as a function of normalized time identified here for (a) type I for with the control parameters $r = 256$ and $\omega_0 = 3.74$, (b) type III with the parameters $r = 28$, $\omega_0 = 4.8$, (σ =10, $\sigma_{\mathrm{m}} = 1$, $b = 8/3$).

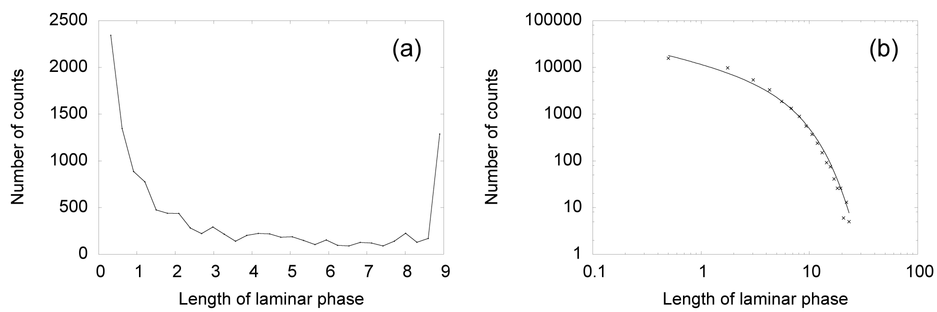

Fig. 3. Distribution of the lengths of laminar phases for (a) $r = 256$, $\omega_0 = 3.74$, $\sigma_{\mathrm{m}} = 1$. The distribution is characterized by U shape and finite value of maximum length of laminar phase, which is characteristic for type I intermittency, Eq. (5), (b) for type III intermittency for small $\tau \to 0$ is consistent with power-law dependence, $P(\tau) \sim \tau^{-3/2}$, observed for fully developed turbulence, while for large $\tau \to \infty$ follows exponential behavior, $P(\tau) \sim \mathrm{e}^{-2\varepsilon\tau}$, predicted by self-organized criticality models, taken from (Macek and Strumik, 2010, 2014).

piecewise numerical solution of Eqs. (1)–(4) is a set of points in the phase space, thus based on the average distance between the points and their nearest neighbors found in the laminar pattern we can identify laminar phases.

Here in Fig. 3 we show the probability distribution of the laminar time intervals τ for our model of Eqs. (1)-(4), where a nontrivial nonlinear dependence is well approximated by the theoretical formulae for type I intermittency

$$P(\tau) = \frac{\varepsilon}{2c}\Big\{1 + \tan^2\Big\{\arctan[\frac{c}{(\varepsilon u)^{1/2}}]\Big\} - \tau(\varepsilon u)^{1/2}\Big\} \tag{5}$$

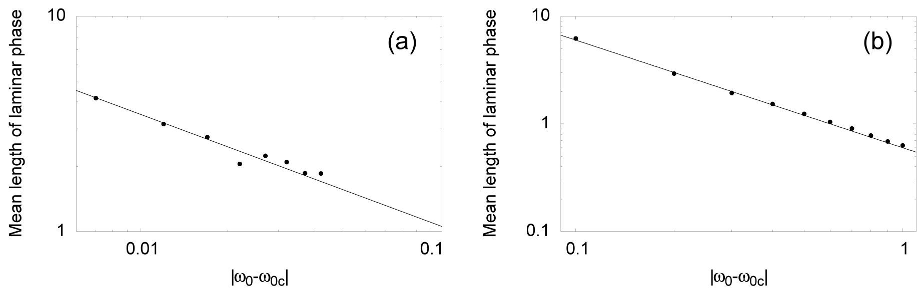

Fig. 4. Scaling of the mean length of the laminar phase with control parameter $\varepsilon = |\omega_0 - \omega_{0c}|$ (for the cases shown in Fig. 3 (a) and (b)), where ω_{0c} is a critical value at which intermittency appears: (a) for type I intermittency the dependence resulting from computations (circles) can be approximated by $\propto \varepsilon^{-1/2}$ function (solid line), (b) for type III intermittency $\propto \varepsilon^{-1}$, cf. (Macek and Strumik, 2014)

where $\varepsilon = |\omega_0 - \omega_{0c}|$ is the difference between the actual value of the control parameter and its critical value for the onset of intermittency (c is the maximum value of the variable in the laminar region, u is another fitting parameter) [14]. Similarly, type III intermittency is given also in Ref. [14]

$$P(\tau) \sim \frac{\varepsilon^{3/2} \mathrm{e}^{4\varepsilon\tau}}{(\mathrm{e}^{4\varepsilon\tau} - 1)^{3/2}}. \tag{6}$$

In Ref. [8] some solutions (for $r = 28$, $\omega_0 = 4.8$, $\sigma = 10$, $\sigma_\mathrm{m} = 1$, $b = 8/3$) of the dynamical system of Eqs. (1)-(4) have been discussed as examples of type III intermittent behavior. It is well known that the classical Lorenz system exhibits type I intermittency transition from periodic to chaotic dynamics for the value of control parameter $r \approx 166.06$. In fact, in the generalized Lorenz model we have identified a branch of periodic-chaotic boundary originating from this point for $\omega_0 = 0$ in the parameter plane. When the magnetic field is taken into account, type I intermittency occurs along this branch, e.g., for $r = 256$, $\omega_0 \approx 3.74$, $\sigma_\mathrm{m} = 1$, which is illustrated in Fig. 3 (a), showing characteristic U-shape of the distribution of laminar phases [9]. For this type of intermittency the maximum length of laminar phase has some finite value. Moreover, as is shown in Fig. 4 (a) in this case we observe another characteristic attributes of the type I intermittency, namely scaling of the mean length of laminar phase with control parameter $\propto \varepsilon^{-1/2}$. One should also note that this functional dependence for type III intermittency as is shown in Fig. 3 (b) for small τ ($\tau \to 0$) is consistent with power-law dependence, $P(\tau) \sim \tau^{-3/2}$, observed for fully developed turbulence [2]. However, for large τ ($\tau \to \infty$) follows exponential behavior, $P(\tau) \sim \mathrm{e}^{-2\varepsilon\tau}$, predicted by self-organized criticality models.

4 Conclusions

For some values of control parameters near the border between periodic and chaotic solutions, but still in chaotic regime, we have observed types I and

III intermittent behavior of the system, which provide mechanisms of release of the bursts of kinetic and magnetic energy. We have discussed the types of bifurcations leading to intermittency. Naturally, these transitions from regular to irregular behavior result from nonlinearity. From the point of view of the theory of dynamical systems, those phenomena are owing to the disappearance of the fixed points or due to change in their their stability. It would be interesting to look for the remaining basic type II intermittency and the respective Hopf bifurcation in our model of hydromagnetic convection.

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