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From chaotic motion to Brownian motion: A survey and some connected problems

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Abstract. In this paper we shall refer to the passing from chaotic motion to Brownian motion. To this end a review of some aspects concerning the Markovian nature of the Brownian path is presented. We discuss about some interesting results regarding to the 3-dimensional Brownian motion in connection with the Markov process in a generalized sense and the k-dimensional Brownian motion in connection with the Dirichlet problem. Then, we shall refer to some special connected studies.
Keywords: stochastic calculus, Markov processes, Markov property, Brownian motion, convergence.

1 Introduction

Let us imagine a chaotic motion of a particle of colloidal size immersed in a fluid. Such a chaotic motion of a particle is called, usually, *Brownian motion* and the particle which performs such a motion is referred to as a *Brownian particle*. Such a chaotic perpetual motion of a Brownian particle is the result of the collisions of particle with the molecules of the fluid in which there is.

But this particle is much bigger and also heavier than the molecules of the fluid which it collide, and then each collision has a negligible effect, while the superposition of many small interactions will produce an observable effect.

On the other hand, for a Brownian particle such molecular collisions appear in a very rapid succession, their number being enormous. For a so high frequency, evidently, the small changes in the particle's path, caused by each single impact, are too fine to be observable. For this reason the exact path of the particle can be described only by statistical methods.

Used especially in Physics, Brownian motion is of ever increasing importance not only in Probability theory but also in classical Analysis. Its fascinating proper-ties and its far-reaching extension of the simplest normal limit theorems to functional limit distributions acted, and continue to act, as a catalyst in random ana-lysis. As some authors remarks too, the Brownian motion reflects a perfection that seems closer to a law of nature than to a human invention.

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Brownian motion was frequently explained as due to the fact that particles were alive.

We remind that Poincaré thought that it contradicted the second law of Thermodynamics.

Today we know that this motion is due to the bombardament of the particles by the molecules of the medium. In a liquid, under normal conditions, the order of magnitude of the number of these impacts is of 10^{20} per second. It is only in 1905 that kinetic molecular theory led Einstein to the first mathematical model of Brownian motion. He began by deriving its possible existence and then only learned that it had been observed.

A completely different origin of mathematical Brownian motion is a game theoretic model for fluctuations of stock prices due to L. Bachélier from 1900.

In the sequel we shell refer shortly to his vision. At the same time we shall discuss some aspects regarding the Markovian nature of the Brownian path, the 3-dimensional Brownian motion in connection with a Markov process in a generalized sense and the extension to the k-dimensional Brownian motion. Finally, we shall refer shortly to some special connected studies.

2 The Markovian nature of the Brownian path

In his thesis (*Théorie de la spéculation*, Ann. Sci. École Norm. Sup. 17, 21-86, 1900) Bachélier found some solutions of the type $\psi(x)$. He derived the law governing the position of a single grain performing a 1-dimensional Brownian motion starting at $a \in \mathbf{R^1}$ at time $t = 0$:

$$P_a[x(t) \in db] = g(t, a, b)db \qquad (t, a, b) \in (0, +\infty) \times \mathbf{R^2}, \tag{1}$$

where g is the source (Green) function

$$g(t, a, b) = \frac{e^{-\frac{(b-a)^2}{2t}}}{\sqrt{2\pi t}} \tag{2}$$

of the problem of heat flow:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial a^2} \qquad (t > 0). \tag{3}$$

Bachélier also pointed out the Markovian nature of the Brownian path expressed in

$$\begin{gathered} P_a[a_1 \le x(t_1) < b_1,\, a_2 \le x(t_2) < b_2,\, \cdots,\, a_n \le x(t_n) < b_n] = \\ = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} g(t_1,\, a,\, \xi_1)\, g(t_2 - t_1, \xi_1, \xi_2) \cdots \\ \cdots g(t_n - t_{n-1},\, \xi_{n-1},\, \xi_n)\, d\xi_1\, d\xi_2 \cdots d\xi_n, \quad 0 < t_1 < t_2 < \cdots t_n \end{gathered} \tag{4}$$

and used it to establish the law of maximum displacement

$$P_0\left[\max_{s \le t} x(s) \le b\right] = 2\int_0^b \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}}\, da \qquad t > 0,\ b \ge 0. \tag{5}$$

It is very interesting that A. Einstein, in 1905, also derived (1) from statistical mechanical considerations and applied it to the determination of molecular diameters (see his work *Investigations on the theory of the Brownian movement*, New York, 1956).

The Brownian motion can be defined as follows

Definition 21 *A continuous-time stochastic process* $\{B_t \,|\, 0 \le t \le T\}$ *is called a "standard Brownian motion" on* $[0, T)$ *if it has the following four properties:*

i $B_0 = 0$.

ii *The increments of* B_t *are independent; that is, for any finite set of times* $0 \le t_1 < t_2 < \cdots < t_n < T$, *the random variables*

$$B_{t_2} - B_{t_1},\, B_{t_3} - B_{t_2},\, \cdots, B_{t_n} - B_{t_{n-1}}$$

are independent.

iii *For any* $0 \le s \le t < T$ *the increment* $B_t - B_s$ *has the normal distribution with mean* 0 *and variance* $t - s$.

iv *For all* ω *in a set of probability one,* $B_t(\omega)$ *is a continuous function of* t.

The Brownian motion can be represented as a random sum of integrals of orthogonal functions. Such a representation satisfies the theoretician's need to prove the existence of a process with the four defining properties of Brownian motion, but it also serves more concrete demands. Especially, the series representation can be used to derive almost all of the most important analytical properties of Brownian motion. It can also give a powerful numerical method for generating the Brownian motion paths that are required in computer simulation.

3 In short about the Markov process in the generalized sense

A Markov process can be defined as follows:

Definition 31 *A Markov process is a system of stochastic processes*

$$\{X_t(\omega), t \in T, \omega \in (\Omega, K, P_a)\}_{a \in S},$$

that is for each $a \in S$, $\{X_t\}_{t \in S}$ *is a stochastic process defined on the probability space* (Ω, K, P_a).

But it is not difficult to observe that a definition of a *Markov process* as in Definition 31 not correspond to many processes that are of a real interest. For this reason it is useful to obtain an extension of this notion. Such an extended notion has been proposed by K. Itô ([6]) and we shall refer to it shortly.

Let E be a separable Banach space with real coefficients and norm $||\cdot||$ and let also $L(E, E)$ be the space of all bounded linear operators $E \longrightarrow E$. It can be observed that $L(E, E)$ is a linear space.

Definition 32 *The collection of stochastic processes*

$$X = \{X_t(\omega) \equiv \omega(t) \in S,\, t \in T,\, \omega \in (\Omega, K, P_a)\}_{a \in S}$$

is called a "Markov process" if the following conditions are satisfied:

1) *the "state space" S is a complete separable metric space and $K(S)$ is a topolo-gical σ-algebra on S;*
2) *the "time internal" $T = [0, \infty)$;*
3) *the "space of paths" Ω is the space of all right continuous functions $T \longrightarrow S$ and K is the σ-algebra $K[X_t : t \in T]$ on Ω;*
4) *the probability law of the path starting at a, $P_a(H)$, is a probability measure on (Ω, K) for every $a \in S$ which satisfy the following conditions:*
4a) $P_a(H)$ *is $K(S)$-measurable in a for every $H \in K$;*
4b) $P_a(X_0 = a) = 1$;
4c) $P_a(X_{t_1} \in E_1, \cdots, X_{t_n} \in E_n) =$
$$\int \ldots \int_{a_i \in E_i} P_a(X_{t_1} \in da_1) P_{a_1}(X_{t_2 - t_1} \in da_2) \ldots$$
$$\ldots P_{a_{n-1}}(X_{t_n - t_{n-1}} \in da_n) \qquad \textit{for} \quad 0 < t_1 < t_2 < \ldots < t_n.$$

According to Definition 32, X will be referred as a *Markov process in the generalized sense.*

Now let X be a Markov process in a generalized sense and let us denote by $\mathbf{B}$(S) the space of all bounded real $K(S)$-measurable functions. Also let us consider a function $f \in \mathbf{B}(S)$.

It is supposed that

$$E_a\Big(\int_0^\infty |f(X_t)|dt\Big) \tag{6}$$

is bounded in a. Therefore

$$Uf(a) = E_a\Big(\int_0^\infty f(X_t)dt\Big) \tag{7}$$

is well-defined and is a bounded $K(S)$-measurable function of $a \in S$.

The Uf is called *the potential* of f with respect to X. Having in view that $Uf = \lim_{\alpha \downarrow 0} R_\alpha f$, it is reasonable to write R_0 instead of U. Based on this fact, $R_\alpha f$ will be called *the potential of order* α of f.

Remark 1. It is useful to retain that $R_\alpha f \in \mathbf{B}(S)$ for $\alpha > 0$; and generally $f \in \mathbf{B}(S)$ while $R_0 f (= Uf) \in \mathbf{B}(S)$ under the condition (6).

Now the name *potential* is justified by the following theorem on *the 3-dimensional Brownian motion*

Theorem 31 *Let X be the 3-dimensional Brownian motion. If $f \in \mathbf{B}(S)$ has compact support, then f satisfies (6) and*

$$Uf(a) = \frac{1}{2\pi} \int_{R^3} \frac{f(b)db}{|b-a|} = \frac{1}{2\pi} \times Newtonian\ potential\ of\ t. \tag{8}$$

Let us denote by D a bounded domain in $R^n, n \geq 1$.

Definition 33 *A function g is called "harmonic" in D if g is C^∞ in D and if $\Delta g = 0$ (where C^∞ is the class of functions differentiable infinitely many times).*

Now let f be a continuous function defined on the boundary ∂D and let us denote by X a k-dimensional Brownian motion defined as follows

Definition 34 *The k-dimensional Brownian motion is defined on $S = R^k$ by the equality*

$$p_t(a, db) = (2\pi t)^{-\frac{k}{2}} e^{-\frac{|b-a|^2}{2t}} db = N_t(b-a)db,$$

where $|b-a|$ is the norm of $b-a$ in R^k.

Given a k-dimensional Brownian motion X, if there exists a solution g for the Dirichlet problem (D, f)[1], then

$$g(a) = E_a(f(X_\lambda)), \tag{9}$$

where $\lambda \equiv \lambda_D$ = exit time from D (that is to say $\lambda_D = inf\{t > 0 : X_t \notin D\}$, the hitting time of D^C).

In this context an interesting result is given in the following theorem

Theorem 32 *If D is a bounded domain and g is a solution of the Dirichlet problem (D, f), then*

$$g(a) = E_a(f(X_\lambda))$$

where $a \in D$ and $\lambda = \lambda_D$.

On the other hand, the Dirichlet problem (D, f) has a solution if ∂D is smooth as it is prooved in the following theorem

Theorem 33 *If ∂D is smooth, then*

$$g(a) = E_a(f(X_\lambda)),$$

where $\lambda = \lambda_D$ = exit time from D, is the solution of the Dirichlet problem (D, f).

Note 31 *The expression "∂D is smooth" means that ∂D has a unic tangent plane at each point x of ∂D and the outward unit normal of the tangent plane at x moves continously with x.*

[1] The Dirichlet problem (D, f) is to find a continuous function $g = g_{D,f}$ on the closure $\overline{D} \equiv D \cup \partial D$ such that g is harmonic in D and $g = f \circ g\, \partial D$.

4 A general survey of some special connected studies

Bachélier was unable to obtain a clear picture of the Brownian motion and his ideas were unappreciated at that time. This because a precise definition of the Brownian motion involves a measure on the path space, and it was not until 1909 when É. Borel published his classical memoir on Bernoulli trials (*Les probabilités dénombrables et leurs applications arithmétique* Rend. Circ. Mat. Palermo 27, 1909, 247-271.

As soon as the ideas of Borel, Lebesgue and Daniell appeared, it was possible to put the Brownian motion on a firm mathematical foundation and this was achived by N. Wiener in 1923 (*Differential space*, J. Math. Phis. 2,1923, 131-174).

Many researchers were fascinated by the great beauty of the theory of Brownian motion and many results have been obtained in the last decades. As for example, among other things, in *Diffusion processes and their sample paths* by K. Itô and H.P. McKean, Jr., in *Theory and applications of stochastic differential equations* by Z. Schuss, or in *Stochastic approximation* by M.T. Wasan as in *Stochastic calculus and its applications to some problems in finance* by J.M. Steele. In this context one can consider also our book *Aspects of convergence and approximation in random systems analysis.*

As we have already emphasized a rigorous definition and study of (mathematical) Brownian motion requires measure theory.

Consider the space of *continuous* path $w : t \in [0, +\infty) \to \mathbf{R^1}$ with coordinates $x(t) = w(t)$ and let β be the smallest Borel algebra of subsets B of this path space which includes all the simple events

$$B = (w \, : \, a \leq x(t) < b), \;\; (t \geq 0, \, a < b).$$

Wiener established the existence of non-negative Borel measures $P_a(B)$, ($a \in \mathbf{R}^1, B \in \beta$) for which (4) holds. Among other things, this result attaches a precise meaning to Bachélier's statement that *the Brownian path is continuous.*

Paul Lévy (*Sur certain processus stochastiques homogènes*, Compositio Math. 7, 1939, pp. 283-339) found another construction of the Brownian motion and also gave a profound description of the fine structure of the individual Brownian path[2].

Lévy's results with several complements due to D.B. Ray (*Sojourn times of a diffusion process*, IJM 7, 1963, 615-630) and K. Itô & H.P. McKean Jr. (*Diffusion processes and their Sample Path*, Springer-Verlag Berlin heidelberg, 1956) are of a special attention to the standard Brownian *local time* (*la measure du voisinage* of P. Lévy):

$$\tau(t, a) = lim_{b \downarrow a} \frac{measure(s : a \leq x(s) < b, \, s \leq t)}{2(b-a)}. \tag{10}$$

Given a Sturm-Liouville operator

$$\mathcal{D}(c_2/2)D^2 + c_1 D, \;\; c_2 > 0$$

[2]P. Lévy, *Processus stochastiques et mouvement brownien*, Paris, 1948

on the line, the source (Green) function $p = p(t, a, b)$ of the problem

$$\frac{\partial u}{\partial t} = \mathcal{D}u, \quad t > 0 \tag{11}$$

share with the Gauss kernel g of (2) the properties:
(a) $0 \le p$
(b) $\int_{R^1} p(t, a, b)db = 1$
(c) $p(t, a, b) = \int_{R^1} p(t - s, a, c)p(s, c, b)dc, \quad t > s > 0.$

Soon after the publication of Wiener's monograph (*Generalized harmonic ana-lysis*, Acta Math. 5, 1930, 117-258), the associated stochastic motions (diffusions) analogous to the Brownian motion ($\mathcal{D} = D^2/2$) made their debut. At a later date (1946) K. Itô (*On a stochastic integral equation*, Proc. Japan acad. 22, 1946, 32-35) proved that if

$$|c_1(b) - c_1(a)| + |\sqrt{c_2(b)} - \sqrt{c_2(a)}| < constant \times |b - a|, \tag{12}$$

then the motion associated with

$$\mathcal{D} = (c_2/2)D^2 + c_1 D$$

is identical in law to the "continuous" solution of

$$a(t) = a(0) + \int_0^t c_1(a)ds + \int_0^t \sqrt{c_2(a)}db \tag{13}$$

where b is a standard Brownian motion.

W. Feller took to lead in the next development. Given a Markovian motion with sample paths $w : t \to x(t)$ and probabilities $P_a(B)$ on a linear interval Q, the operators

$$H_t : f \to \int P_a[x(t) \in db]f(b) \tag{14}$$

constitute a *semi-group* :

$$H_t = H_{t-s}H_s, \quad t \ge s \tag{15}$$

and as E. Hille (*Represenation of one-parameter semi-groups of linear tansformations*, PNAS 28, 1942, 175-178) and K. Yosida (*On the differentiability and the representation of one-parameter semi-group of linear operators*, J. Math. Soc. Japan 1, 1948, 15-21) proved,

$$H_t = e^{t\mathcal{D}}, \quad t > 0 \tag{16}$$

with a suitable interpretation of the exponential, where $\mathcal{D}$ is the so-called *generator*.

We mention again the name of D. Ray to emphasize that he proved (*Stationary Markov processes with continuos path*, TAMS, 82, 1956, pp. 452-493) that if the motion is *strict Markov* (i.e. if it *starts afresh* at certain stochastic (Markov) times including that passage times $m_a = \min(t : x(t) = a)$, etc.), then the so-called *generator* $\mathcal{D}$ is *local* if and only if the motion has *continuous* sample paths, substantiating a conjecture of W. Feller.

Then by combining this with some other Feller's papers as

• W. Feller, *The paraboloc differential equations and the associated semi-groups of tansformaions*, AM 55, 1952, 468-519;

• W. Feller, *The general diffusion operator and positivity preserving semi-groups in one dimension*, AM 60, 1954, 417-436;

• W. Feller, *On second order differential operators*, AM 61, 1955, 90-105;

• W. Feller, *Generalized second order differential operators and their lateral conditions*, IJM 1, 1957, 456-504,

it is implied that the generator of a strict Markovian motion with continuous paths (diffusion) can be expressed as a *differential operator*

$$(\mathcal{D}u)(a) = \lim_{b \downarrow a} \frac{u^+(b) - u^+(a)}{m(a,b)}, \tag{17}$$

where m is a non-negative Borel measure positive on open intervals and, with a change of scale

$$u^+(a) = \lim_{b \downarrow a} (b-a)^{-1} [u(b) - u(a)],$$

except of certain singular points where $\mathcal{D}$ degenerates to a differential operator of degreee ≤ 1.

Finally we remark that E.B. Dynkin (*Continous one-dimensional Markov processes*, Dokl. Akad. Nauk SSSR, 105, 1955, 405-408) also arrived at the idea of a stict Markov process. He derived an elegant formula for $\mathcal{D}$ and used it to make a simple (proba-bilistic) proof of Feller's expression for $\mathcal{D}$.

At the same time we consider that the papers of R. Blumenthal - *An extended Markov property*, TAMS 85, 1957, 52-72, and G. Hunt - *Some theorems concerning Brownian motion*, TAMS 81, 1956, 294-319, as well as the monographs of E.B. Dynkin - *Principles of the theory of Markov random processes*, Moskow-Leningrad, 1959; and *Markov processes*, Moskow, 1963, must also to be mentioned in such a connection.q

Remark 2. Many other details regarding to the topics just discussed, proofs and some related problems can be found in [6], [5], [1], [4], [21], [10], [22], [9], [15], [13].

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