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# A note on the bifurcation point of a randomized Fibonacci model

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**Abstract.** In a recent paper, a modified randomized Fibonacci model was presented, assuming that the number of direct offsprings of each ancestor is a Bernoulli random variable. In this work a question posted in the referred paper on the modified randomized Fibonacci model is answered and, in addition, the model is studied from a discrete dynamical system approach. In particular, the local qualitative properties of the unique bifurcation point that emerges in the unit interval in the Randomized Fibonacci model are presented. The work ends with a study of the long-term behaviour of a multi species ruled by the modified randomized Fibonacci model.
**Keywords:** Randomized Fibonacci model, Bernoulli offsprings, Local stability, Bifurcation point.

## 1 Introduction

In a groundbreaking paper by Brilhante, Gomes and Pestana [5], the authors used branching process to introduced a more realistic *Fibonacci-type* model. The idea is to include in the classical Fibonacci model the effects of some random phenomena that occur naturally in life, and that limit population growth, as limited resources, existence of predators, of illnesses, among others. These phenomena are not considered in the original Fibonacci model that allows a population to grow without limits, since its members never die and never stop to reproduce themselves.

In the above mentioned work, the authors follow the idea that each ancestor can produce direct offspring only in the first two consecutive reproducing periods. Starting with only one ancestor, single or couple (according to the reproduction characteristics of the species), in each of the two initial reproducing periods each unit produces $X \frown Bernoulli(p)$ offspring, with $p \in (0,1)$, and is removed from the process after the second reproduction. Notice that

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it is assumed that each offspring becomes an ancestor in the following step, behaving exactly in the same fashion.

In the following, $Z_1$ represents the number of units of the population in the instant when the initial ancestor is removed from it (this moment will be called the *first step of the process*):

$$Z_1 = \begin{cases} 0 & 1 & 2 & 3 \\ (1-p)^2 & p(1-p)(2-p) & 2p^2(1-p) & p^3 \end{cases} \tag{1}$$

In the second step of the process, the number of units of the population, $Z_2$, is given by

$$Z_2 = \sum_{k=1}^{Z_1} Z_1^{(k)},$$

where $Z_1^{(k)}$ is the number of the direct offspring of the $k$th unit of the population of the first step of the process, minus one, since, once more, this unit is removed from the process (notice that if $Z_1$ is null, then $Z_2$ is also null). Here we admit that the random variables $Z_1^{(k)}$, $k = 1, 2, ...,$ are independent and identically distributed (iid), and independent from $Z_1$. In the $n$th step, the number of units of the population is given by

$$Z_n = \sum_{k=1}^{Z_1} Z_{n-1}^{(k)},$$

where $Z_{n-1}^{(k)}$ is the number of the offspring of the $k$th unit of the population of the first step of the process at the $(n-1)$th step, minus the ancestors of the previous steps. Again, the random variables $Z_{n-1}^{(k)}$, $k = 1, 2, ...,$ are iid, and independent from $Z_1$. The probability generating function (pgf) of $Z_1$ is given by, for $x \in \mathbb{R}$,

$$\mathcal{G}(x) = \mathcal{G}_1(x) = E\left(x^{Z_1}\right) = (1-p)^2 + p(1-p)(2-p)x + 2p^2(1-p)x^2 + p^3x^3, \tag{2}$$

and the pgf of $Z_n$ is given by

$$\begin{aligned} \mathcal{G}_n(x) &= E\left(x^{Z_n}\right) = E\left(x^{\sum_{k=1}^{Z_1} Z_{n-1}^{(k)}}\right) = \sum_{i=0}^{3} E\left(x^{\sum_{k=1}^{Z_1} Z_{n-1}^{(k)}} \,\middle|\, Z_1 = i\right) P(Z_1 = i) \\ &= \sum_{i=0}^{3} \left[\prod_{k=1}^{i} E\left(x^{Z_{n-1}^{(k)}}\right)\right] P(Z_1 = i) = \sum_{i=0}^{3} \left[\prod_{k=1}^{i} \mathcal{G}_{n-1}(x)\right] P(Z_1 = i) \\ &= \sum_{i=0}^{3} \left[\mathcal{G}_{n-1}(x)\right]^i P(Z_1 = i), \\ &= \mathcal{G}_1(\mathcal{G}_{n-1}(x)), \end{aligned}$$

i.e.,

$$\mathcal{G}_n(x) = \mathcal{G}_1(\mathcal{G}_{n-1}(x)), \; n = 2, 3, ..., \tag{3}$$

which means that the function $\mathcal{G}_n$ is recursively defined having as initial condition (2). On the other hand, the probability that the population is extinct at some step $n = 1, 2, ...$ is given by

$$x_n = P(Z_n = 0) = \mathcal{G}_n(0).$$

Using (2) we obtain, for $n = 2, 3, ...$,

$$x_n = \mathcal{G}_1(x_{n-1}), \tag{4}$$

a recursive relation that determines $x_n$, having in mind that $x_1 = (1-p)^2 \in (0,1)$ (cf. expression (1)). Following Feller's argument in [6], notice that the function $\mathcal{G}_1$ is strictly increasing and therefore $x_1 < x_2 < x_3 < ....$ On the other hand, if $x \in [0,1]$ then $\mathcal{G}_1(x) \in [0,1]$, which means that there exists a limit $x = \lim_{n\to+\infty} x_n$ such that $x \leq 1$ and, from (3), that

$$x = \mathcal{G}_1(x).$$

At the long range, the population will become extinct with probability $x$, a fixed point of $\mathcal{G}_1$.

The expected size of the population at the $n$th step $(n = 2, 3, ...)$ is given by $\mathcal{G}_n^{'}(1)$ and, from (4),

$$E(Z_n) = \mathcal{G}_n^{'}(1) = \mathcal{G}_1^{'}(\mathcal{G}_{n-1}(1))\mathcal{G}_{n-1}^{'}(1).$$

Since

$$\mathcal{G}_1(1) = (1-p)^2 + p(1-p)(2-p) + 2p^2(1-p) + p^3 = 1,$$

then $\mathcal{G}_{n-1}(1) = 1$. Setting $\mu = \mathcal{G}_1^{'}(1)$ we obtain

$$\mathcal{G}_n^{'}(1) = \mu\mathcal{G}_{n-1}^{'}(1)$$

and, consequently, the expected size of the population at the $n$th step, $n = 1, 2, ...$, is given by

$$E(Z_n) = \mu^n,$$

with

$$\mu = \mathcal{G}_1^{'}(1) = p(1-p)(2-p) + 4p^2(1-p) + 3p^3 = p(p+2).$$

This means that when $\mu < 1$, the expected size of the population converges to zero, and when $\mu > 1$, the expected size of the population converges to infinity, as $n$ grows . Since $p \in (0,1)$, $p(p+2) < 1 \Rightarrow p \in \left(0, \sqrt{2}-1\right)$ and $p(p+2) > 1 \Rightarrow p \in \left(\sqrt{2}-1, 1\right)$.

When $\mu = 1$, i.e., when $p = \sqrt{2}-1$, we have a special case that needs a deep attention from a discrete dynamical system point of view. To do so, in the next two sections some basic but powerful tools that allow us to know the dynamics in this specific value of $p$ will be presented. The expected size of the population is, in this case, 1.

## 2 Local stability of the modified Fibonacci model: a discrete dynamical system approach

The original problem presented by Leonardo Pisano, better known as Fibonacci, in 1202 in the book “Liber Abaci” [12], is usually found in the books on this area of difference equations. The Fibonacci model has motived many studies, some of which include variants of the original problem, such as the previous cited work of the area of statistics. Many other variants can be found also in the area of pure discrete dynamical systems (cf., e.g., [9]).

In this section we study the modified Fibonacci model defined in the previous section from a discrete dynamical system point of view. At the same time we provide a review of the basic results in this field in order to clarify the readers that are not familiar with these.

Let us consider the difference equation given by

$$x_{n+1} = \mathcal{G}\left(x_n\right) \tag{5}$$

with the map $\mathcal{G}$ given by (2), i.e.,

$$x_{n+1} = (1-p)^2 + p(1-p)(2-p)x_n + 2p^2(1-p)x_n^2 + p^3 x_n^3, \tag{6}$$

$x_n \geq 0$ for $n = 0, 1, 2, 3, \ldots$. The range of the parameter $p$ is here extended to the positive real numbers, $p > 0$ (instead of having $p \in (0,1)$).

Consider an interval $I \subseteq \mathbb{R}$ and a map $f : I \to I$. A point $x^* \in \mathbb{R}$ is said to be a fixed point (or equilibrium point) of $f$ if $f(x^*) = x^*$, and given $x_0 \in \mathbb{R}$ , we define its orbit $O(x_0)$ as the set of points $O(x_0) = \{x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots\}$, where $f^2 = f \circ f, f^3 = f \circ f \circ f, \ldots$. One of the main objectives of the theory of discrete dynamical systems and, in particular, of the stability theory, is the study of the behavior of orbits near fixed points, i.e., the behavior of solutions of difference equations when the starting points are near equilibrium points. Hence, a basic definition in this field is needed. Let $\mathbb{N}$ denote the set of nonnegative integers.

**Definition 1 (Local stability).** Let $f : I \to I$ be a map and $x^*$ be a fixed point of $f$, where $I$ is an interval of real numbers. Then

1. $x^*$ is said to be locally stable if for any $\epsilon > 0$ there exits $\delta > 0$ such that for all $x_0 \in I$ with $|x_0 - x^*| < \delta$ we have $|f^n(x_0) - x^*| < \epsilon$ for all $n \in \mathbb{N}$. Otherwise, the fixed point $x^*$ will be called unstable.
2. $x^*$ is said to be attracting if there exists $\eta > 0$ such that $|x_0 - x^*| < \eta$ implies $\lim\limits_{n\to\infty} f^n(x_0) = x^*$.
3. $x^*$ is said to be locally asymptotically stable if it is both stable and attracting. If in the previous item $\eta = \infty$, then $x^*$ is said to be globally asymptotically stable.

One of the most effective graphical iteration methods to depict stability of fixed points is the cobweb diagram (also known as stair-step diagram). For more details about the notions of stability and the exploration of concrete examples using cobweb diagrams we refer two works of Elaydi, [2] and [3].

However, cobweb diagrams are not the most efficient tool to study local stability. There exists a simple but powerful criterion for knowing the local stability of fixed points. We may divide the fixed points into two categories: hyperbolic and nonhyperbolic. A fixed point $x^*$ of a map $f$ is said to be hyperbolic if $|f'(x^*)| \neq 1$. Otherwise, it is nonhyperbolic. The following theorem is well known in the theory of discrete dynamical systems and may be found in any book on discrete dynamical systems.

**Theorem 1 ([3], page 25).** *Let $x^*$ be a hyperbolic fixed point of a map $f$, where $f$ is continuous differentiable at $x^*$. The following statements hold true:*

1. *If $|f'(x^*)| < 1$, then $x^*$ is locally asymptotically stable.*
2. *If $|f'(x^*)| > 1$, then $x^*$ is unstable.*

The stability criteria for nonhyperbolic fixed points are more complex and are summarized in the following theorem (for a complete classification of nonhyperbolic fixed points and a definition of semi-stability cf. [3], pages 33-35). Before presenting the criteria we introduce the notion of Schwarzian derivative.

**Definition 2 (Schwarzian derivative).** The Schwarzian derivative, $Sf$, of a function $f$, is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

In particular, when $f'(x^*) = -1$, we have $Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2$.

**Theorem 2 ([3], pages 28-30).** *Let $x^*$ be a fixed point of a map $f$ and $f'$, $f''$ and $f'''$ be continuous at $x^*$.*

1. *Let $f'(x^*) = 1$.*
   (a) *If $f''(x^*) > 0$, then $x^*$ is unstable but semi-stable from the left.*
   (b) *If $f''(x^*) < 0$, then $x^*$ is unstable but semi-stable from the right.*
   (c) *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then $x^*$ is unstable.*
   (d) *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then $x^*$ is locally asymptotically stable.*
2. *Let $f'(x^*) = -1$.*
   (a) *If $Sf(x^*) < 0$, then $x^*$ is locally asymptotically stable.*
   (b) *If $Sf(x^*) > 0$, then $x^*$ is unstable.*

Now, let us turn our attention to the difference equation given by (5). The fixed points of $\mathcal{G}$ are the solutions of the equation $x = \mathcal{G}(x)$, i.e., for $p > 0$

$$\begin{aligned}
x_1^* &= 1, \\
x_2^* = x_2^*(p) &= \frac{p(p-2) + \sqrt{p(4-4p+p^3)}}{2p^2}, \\
x_3^* = x_3^*(p) &= \frac{p(p-2) - \sqrt{p(4-4p+p^3)}}{2p^2}.
\end{aligned}$$

The domain of $\mathcal{G}$ has been restricted to $(0,+\infty)$ and $x_3^* < 0$. As a consequence, only $x_1^*$ and $x_2^*$ will be considered.

The derivative of the map $\mathcal{G}$ is given by

$$\mathcal{G}'(x) = p(1-p)(2-p) + 4p^2(1-p)x + 3p^3x^2.$$

Since $\mathcal{G}'(x_1^*) = p(2+p)$, it follows that the fixed point $x_1^* = 1$ is locally asymptotically stable when $|p(2+p)| < 1$, or equivalently $p \in (0, \sqrt{2}-1)$. On the other hand, when $p > \sqrt{2}-1$, $x_1^* = 1$ is unstable. The case when $p = \sqrt{2}-1$ will be studied below.

Now let us study the local stability of the fixed point $x_2^*$. A direct computation leads to

$$\mathcal{G}'\left(x_2^*\right) = \frac{p^3 - 2\sqrt{p^3-4p+4}\sqrt{p} - \sqrt{p^3-4p+4}p^{3/2} - 4p + 6}{2},$$

and shows that $|\mathcal{G}'(x_2^*)| < 1$ if and only if $\sqrt{2}-1 < p < \frac{\sqrt{17}-1}{2}$. Notice that $x_2^*$ is unstable if $p \in \left(0, \sqrt{2}-1\right) \cup \left(\frac{\sqrt{17}-1}{2}, +\infty\right)$.

Now we will study the special case when $p = \sqrt{2}-1$. Notice that for this value of the parameter we have $x^* = x_1^* = x_2^* = 1$, $\mathcal{G}'\left(x^*\right) = 1$ and $\mathcal{G}''\left(x^*\right) = 2\left(\sqrt{2}-1\right) > 0$. Consequently, by Theorem 2 the fixed point $x^*$ is semi-stable from the left. This dynamics is clearly shown in the cobweb diagram presented in Figure 1.

It is important to deeply understand the behavior of the system near $x^* = 1$ for values of parameter $p$ in $(\sqrt{2}-1-\delta, \sqrt{2}-1+\delta)$, with small $\delta > 0$. For $p < \sqrt{2}-1$ there are two fixed points in the system $x_1^*$ and $x_2^*\left(p\right)$, the first of which is locally asymptotically stable, while the second one is unstable. For $p > \sqrt{2}-1$ , $x_1^*$ and $x_2^*\left(p\right)$ are still fixed points, but have exchanged their stability situation: $x_1^*$ is now unstable and $x_2^*\left(p\right)$ is locally asymptotically stable. When $p = \sqrt{2}-1$ the two fixed points (locally asymptotically stable and unstable) "collide", forming a unique fixed point, and this is the only time where this happens, for there will be no more unique fixed points. This behavior is known as bifurcation in the discrete dynamical system. For a general reference in the theory of bifurcation of fixed points cf. Kuznetsov's book [8].

The main types of bifurcation are summarized in Table 1. For a classification of nonhyperbolic fixed points of periodic maps, cf. the work by Elaydi, Luís and Oliveira [4], which generalize the present study.

We will now check the type of bifurcation that occurs when $p = \sqrt{2}-1$, i.e., in $x^* = 1$. Since $\mathcal{G}'\left(x^*\right) = 1$,

$$\frac{\partial \mathcal{G}}{\partial p}(x^*) = 0, \mathcal{G}''(x^*) = 2\left(\sqrt{2}-1\right) \neq 0 \text{ and } \frac{\partial^2 \mathcal{G}}{\partial p \partial x}(x^*) = 2\sqrt{2} \neq 0,$$

we conclude that there is a transcritical bifurcation in $x^* = 1$ (at $p = \sqrt{2}-1$).

At $p = \frac{\sqrt{17}-1}{2}$ we have $x_2^* = x_2^*\left(\frac{\sqrt{17}-1}{2}\right) = \frac{7}{16} - \frac{1}{16}\sqrt{17}$, $\mathcal{G}'(x_2^*) = -1$,

$$\frac{\partial^2 \mathcal{G}}{\partial p \partial x}(x_2^*) = \frac{1}{8}\left(\sqrt{2\left(1+\sqrt{17}\right)} + 10\sqrt{17} - 3\sqrt{34\left(1+\sqrt{17}\right)} - 14\right) \neq 0$$

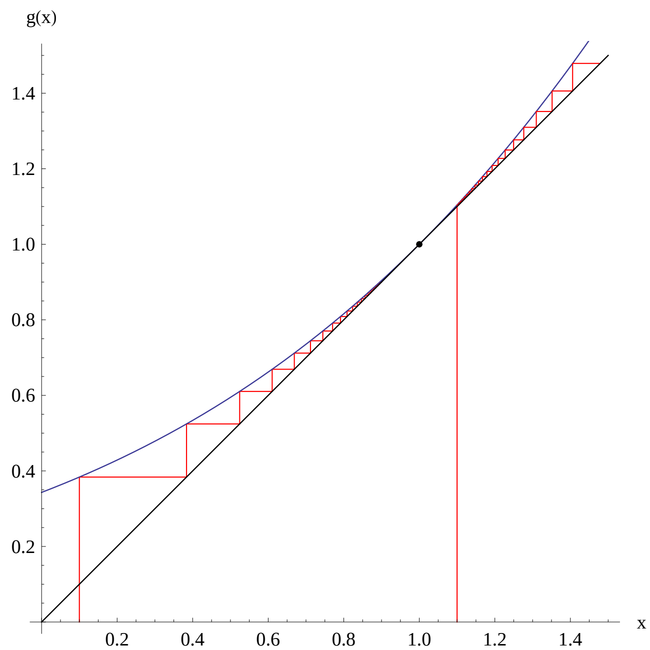


**Fig. 1.** The cobweb diagram of the fixed point $x^* = 1$, in the phase-space diagram, of the map $\mathcal{G}$ at $p = \sqrt{2} - 1$. The fixed point $x^* = 1$ is semi-stable from the left since the orbit of any initial point $x_0 < 1$ converge to $x^* = 1$ while the orbit of a point $x_0$ in the right vicinity of 1, $x_0 \gtrsim 1$, diverge from $x^* = 1$.

| Type of bifurcation in $x^*$ | $f'(x^*)$ | $\frac{\partial f}{\partial p}(x^*)$ | $f''(x^*)$ | $\frac{\partial^2 f}{\partial p \partial x}(x^*)$ | $f'''(x^*)$ | $Sf(x^*)$ |
|---|---|---|---|---|---|---|
| Saddle-node | 1 | $\neq 0$ | $\neq 0$ | | | |
| Transcritical | 1 | 0 | $\neq 0$ | $\neq 0$ | | |
| Pichfork | 1 | 0 | 0 | $\neq 0$ | $\neq 0$ | |
| Period-doubling | -1 | | | $\neq 0$ | | $\neq 0$ |

**Table 1.** The main types of bifurcation for nonhyperbolic fixed points in one-dimensional maps.

and

$$S\mathcal{G}(x_2^*) = \frac{3}{8}\left(\sqrt{17} - 9\right)\left(3\sqrt{2\left(1 + \sqrt{17}\right)} + 3\sqrt{17} - 1\right) \neq 0,$$

which means that there is period-doubling bifurcation in $x_2^*\left(\frac{\sqrt{17}-1}{2}\right)$. The fixed point $x_2^*$ becomes unstable and a new locally stable 2−periodic cycle of the equation $x_{n+1} = g(x_n)$ is born, i.e., the map $g^2 = g \circ g$ has a locally asymptotically stable fixed point. This fixed point of the composition, is in fact the following 2−periodic cycle of the map $g$

$$\left\{\frac{1}{2}\left(-\frac{\sqrt{p^3+2p^2-3p-4}}{p^{3/2}}-\frac{1}{p}+1\right),\frac{1}{2}\left(\frac{\sqrt{p^3+2p^2-3p-4}}{p^{3/2}}-\frac{1}{p}+1\right)\right\}.$$

From this point forward calculations can only be done by numerical methods. If we do so, we will verify that this scenario of period-doubling bifurcation will continue. This behaviour is confirmed by Sharkovsky's theorem. Established by Sharkovsky in 1964, cf. [10], and later translated to English by J. Tolosa in 1995, cf. [11], this result has played a role of paramount importance in the dynamics of one-dimensional maps. First, it is necessary to introduce the Sharkovsky order of the positive integers, that is denoted by the symbol $\rhd$ and defined in the following way:

$$\begin{gathered} 3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \rhd (2n+1)\times 2^0 \rhd \cdots \\ 3\times 2 \rhd 5\times 2 \rhd 7\times 2 \rhd 9\times 2 \rhd \cdots \rhd (2n+1)\times 2^1 \rhd \cdots \\ 3\times 2^n \rhd 5\times 2^n \rhd 7\times 2^n \rhd 9\times 2^n \rhd \cdots \rhd (2n+1)\times 2^n \rhd \cdots \\ \cdots \rhd 2^n \rhd 2^{n-1} \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2^1 \rhd 2^0 \end{gathered}$$

Sharkovsky's theorem can now be stated:

**Theorem 3 ([11]).** *[Sharkovsky's theorem] Let $F : I \to I$ be a continuous map which has a periodic orbit of prime period $k$. Then for any positive integer $l$ that is preceded by $k$ in the Sharkovsky's order, $k \rhd l$, there is a periodic orbit of prime period $l$.*

Note that Theorem 3 is a one-dimensional result and, in general, it is not known if it holds in higher dimensions, although some particular results do exist (cf. e.g. the work of P. Kloeden, [7]).

Solving numerically the equation $\mathcal{G}^3(x^*) = x^*$ when $p = 1.960835$ we find that $x^* \approx 0.489723$ is one of the solutions. This implies that $x^* \approx 0.489723$ is a 3−periodic cycle of the map $\mathcal{G}$. Taking the derivative of the composition we find that $\left(\mathcal{G}^3\right)'(0.489723) \approx 0.858445 < 1$. This implies that the 3−cycle is locally asymptotically stable. Consequently, by Sharkovsky's Theorem the map $\mathcal{G}$ has periodic orbits of all periods.

There exists another way of presenting this bifurcation scenario: plotting a bifurcation diagram in the direct product of the phase and parameter spaces, i.e, in the $(p, x)$−plane, where $x$ represents the limit values of the sequence $x_n(p)$ defined by (5). This diagram is depicted in Figure 2 for the fixed points and the 2−periodic orbits. The solid curves represent the regions of locally asymptotically stability, while the dashed curves represent the regions where instability occurs. The second bifurcation diagram presented in Figure 3 is obtained via simulation.

## 3 Modified randomized Fibonacci model revisited: the bifurcation point

In this section, the solution to an open question posted in [5] at the end of Section 2 is given. In the cited paper the authors compare analytic solutions with numerical results of the equation $x_n = \mathcal{G}_{Z_1}(x_{n-1})$, where $x_n$ is

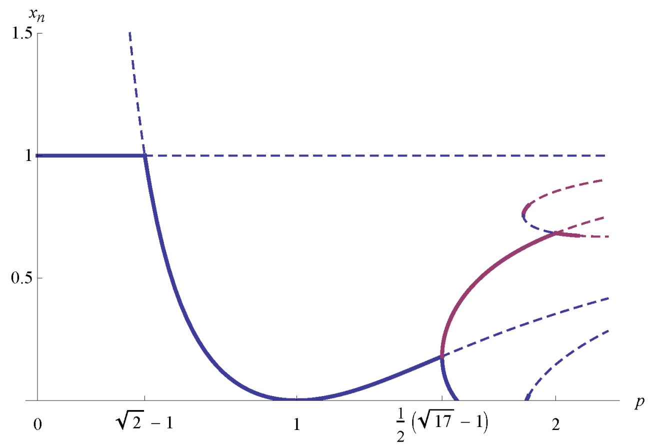


**Fig. 2.** Bifurcation of the fixed points and the appearance of 2−periodic cycles in the $(p, x)$−plane. The solid lines is the region where the fixed points are locally asymptotically stable while in the dashed curves instability occurs.

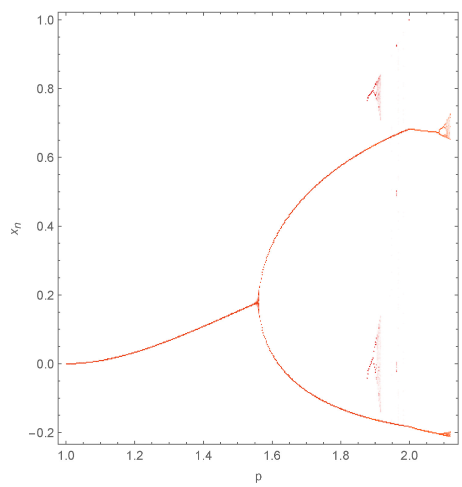


**Fig. 3.** The period-doubling scenario obtained via simulation.

the probability that extinction does occurs at or before the $n$th generation and $x_1 = \mathbb{P}(Z_1 = 0) = (1-p)^2$. The numerical results obtained suggested that the fixed point $x_1^* = 1$ at $p = \sqrt{2} - 1$ is unstable and the authors observed that this instability seemed "quite different in nature from the Feigenbaum bifurcations".

In order to clarify this observation, recall that in the considered randomized Fibonacci model, $X \frown Bernoulli(p)$, with $0 < p < 1$. Consequently, the map $g$ defined in (5) maps the unit square into itself, $g : [0,1] \to [0,1]$.

From the previous section we know that the fixed point $x^* = 1$ at $p = \sqrt{2}-1$ is semi-stable from the left (its dynamics is shown in the cobweb diagram presented in Figure 1). Moreover, it bifurcates transcritically to the second fixed point $x_2^*$. Hence, in this specific point, the instability mentioned above does not occur, since in these special points, where bifurcation occurs, stability does exist. What happened in the cited paper was that the number of iterations used by the authors was not enough to visualize the convergence in the bifurcation point.

Furthermore, since $p \in (0,1)$, there are no periodic points of prime period 2 in $[0,1]$. This can be seen in Figure 2 and confirmed by the following result:

**Theorem 4 ([1]).** *Let $I = [a,b] \subseteq \mathbb{R}$ and $f : I \to I$ be a continuous map. If the equation $f(f(x)) = x$ has no roots, with the possible exception of the roots of the equation $f(x) = x$, then every orbit under the map $f$ converges to a fixed point.*

Finally, since for $p \in (0,1)$ there are no periodic points in $[0,1]$, it is not possible to observe the sequence of the Feigenbaum numbers in the dynamic of the considered randomized Fibonacci model. To observe that phenomenon, it is necessary to have the period-doubling scenario.

## 4 Multi randomized Fibonacci species

In this section, the model studied in the introduction is extended to multi-species. We will consider $k$ ancestors from different species that can coexist in the same habitat and do not compete for resources.

The rule will be the same as before: each ancestor can produce direct off-spring only in the first two consecutive reproducing periods. Starting with only one ancestor of each species, single or couple (according to the reproduction characteristics of the species), in each of the two initial reproducing periods each unit produces

$$X_i \frown Bernoulli(p_i), \quad i = 0, 1, 2, \ldots, k,$$

offspring, with $p_i \in (0,1)$, and is removed from the process after the second reproduction. Notice that it is assumed that each offspring becomes an ancestor in the following step, behaving exactly in the same fashion.

Since there is no competition between species, a natural question arises: what is the effect of the individual expected size in the expected size of the total population? In other words, we know that if $p_i(p_i + 2) < 1$, $i = 1, 2, \ldots, k$, i.e., $p_i \in (0, \sqrt{2} - 1)$, then the expected size of $i$ species converges to zero. Now, we want to know the region, in the in $k-$dimensional space $(p_1, p_2, ..., p_k)$, where the expected size of the total population converges to zero.

Denote $Z_1$ the number of units of the total population when the initial ancestors are removed from the system and $Z_{1,i}$, $i = 1, 2, \ldots, k$, the number of units of the individual populations when each initial ancestor is removed from the system, i.e., for each $i = 1, 2, \ldots, k$ we have

$$Z_{1,i} = \left\{ \begin{array}{cccc} 0 & 1 & 2 & 3 \\ (1-p_i)^2 & p_i(1-p_i)(2-p_i) & 2p_i^2(1-p_i) & p_i^3 \end{array} \right.$$

and

$$Z_1 = \sum_{i=1}^{k} Z_{1,i} = \begin{cases} 0 & 1 & 2 & .. & 3k \\ A_0 & A_1 & A_2 & ... & A_{3k} \end{cases}$$

where $\sum_{i=0}^{3k} A_i = 1$. For instance, when $k = 2$ the numbers $A_i$, $i = 0, 1, ..., 6$ are given by

$$A_i = \sum_{j=0}^{i} P\left(Z_{1,1} = j\right) P\left(Z_{1,2} = i - j\right).$$

At the $n$th step, the number of units of the total population is given by

$$Z_n = \sum_{i=1}^{k} Z_{n,i},$$

where $Z_{n,i}$, $i = 1, 2, \ldots, k$, is the population of the $n$th generation of the $i$th species. Notice that the random variables $Z_{n,i}$, $i = 1, 2, \ldots, k$, are independent. Consequently, the pgf of $Z_n$ is given by

$$G_n = \prod_{i=1}^{k} \mathcal{G}_{n,i},$$

where $\mathcal{G}_{n,i}$ is the pgf of $Z_{n,,i}$ that, recall, satisfies the recurvive equation (3)

$$\mathcal{G}_{n,i}\left(x\right) = \mathcal{G}_{_{1,i}}(\mathcal{G}_{n-1,i}\left(x\right)),\ n = 2, 3, ...,$$

with

$$\mathcal{G}_{1,i}(x) = (1 - p_i)^2 + p_i(1 - p_i)(2 - p_i)x + 2p_i^2(1 - p_i)x^2 + p_i^3 x^3.$$

The probability that the population is extinct at some step $n = 1, 2, ...$is again given by

$$y_n = P\left(Z_n = 0\right) = G_n\left(0\right) = \prod_{i=1}^{k} \mathcal{G}_{_{n,i}}\left(0\right) = \prod_{i=1}^{k} x_{_{n,i}},$$

where $x_{_{n,i}}$is the probability that the $i$th species is extinct at the same step $n$. But

$$x_{n,i} = \mathcal{G}_{_{1,i}}\left(x_{n-1,i}\right), \tag{7}$$

with $x_{1,i} = \left(1 - p_i\right)^2 \in (0, 1)$. Consequently

$$\lim_{n \to +\infty} y_n = \prod_{i=1}^{k} \lim_{n \to +\infty} x_{_{n,i}} = \prod_{i=1}^{k} x_{_i},$$

with $x_i$ such that

$$x_i = \mathcal{G}_{_{1,i}}\left(x_i\right).$$

Naturally, if one species does not become extinct, i.e., if $x_i$ is zero for some $i \in \{1, .., k\}$, then the population does not become extinct neither.

The expected size of the population at the $n$th step $(n = 2, 3, ...)$ is given by

$$E(Z_n) = \sum_{i=1}^{k} E(Z_{n,i}) = \sum_{i=1}^{k} \mu_i^n,$$

with

$$\mu_i = \mathcal{G}_{1,i}^{'}(1) = p_i(p_i + 2),$$

i.e.,

$$E(Z_n) = \sum_{i=1}^{k} p_i^n (p_i + 2)^n.$$

Recall that $p_i \in (0, 1)$, $p_i(p_i + 2) < 1 \Rightarrow p_i \in \left(0, \sqrt{2} - 1\right)$ and $p_i(p_i + 2) > 1 \Rightarrow p_i \in \left(\sqrt{2} - 1, 1\right)$.

Hence, the expected size of the population in long term depends the values of $p_i$, $i = 1, 2, \ldots, k$. We can have:

- Extinction - in the sense that all species will eventually die, i.e., $\lim_{n\to+\infty} E(Z_n) = 0$;
- Exclusion - in the sense that some species will eventually die and others will survive, i.e., $\lim_{n\to+\infty} E(Z_n) = v < k$;
- Coexistence - in the sense that all the $k$ species will remain in the system, i.e., $\lim_{n\to+\infty} E(Z_n) = k$;
- Expansion - in the sense that the expected size will increase without limits, i.e., $\lim_{n\to+\infty} E(Z_n) = +\infty$.

More specifically, the $\lim_{n\to+\infty} E(Z_n)$ is:

- 0 if $p_i < \sqrt{2} - 1$ for all $i \in \{1, ..., k\}$;
- 1 if $p_i = \sqrt{2} - 1$ for some $i \in \{1, ..., k\}$ and $p_j < \sqrt{2} - 1$ for all $j \neq i$, $j \in \{1, ..., k\}$;
- 2 if $p_{i_1} = p_{i_2} = \sqrt{2} - 1$ for some $i_1, i_2 \in \{1, ..., k\}$ and $p_j < \sqrt{2} - 1$ for all $j \in \{1, ..., k\} \setminus \{i_1, i_2\}$;
- ...
- $k$ if $p_i = \sqrt{2} - 1$ for all $i \in \{1, ..., k\}$;
- $+\infty$ if $p_i > \sqrt{2} - 1$ for some $i \in \{1, ..., k\}$.

In Figure 4, the scenarios for two species are presented in the parameter space $(p_1, p_2)$. If $p_1$ and $p_2$ belong to the region $\mathcal{R}_1$, both species will eventually become extinct. If $p_1$ and $p_2$ are in region $\mathcal{R}_2$ there is expansion of species 1 with exclusion of species 2, while in region $\mathcal{R}_3$ species 2 expands with exclusion of species 1. If $p_1$ and $p_2$ are on the segment $l_1$ we verify the exclusion of species 2 while on the segment $l_2$ species 1 will be excluded from the system. On the point C we have the coexistence of both species. Finally, if $p_1$ and $p_2$ are in the region $\mathcal{R}_4$ we eventually observe the expansion of both species.

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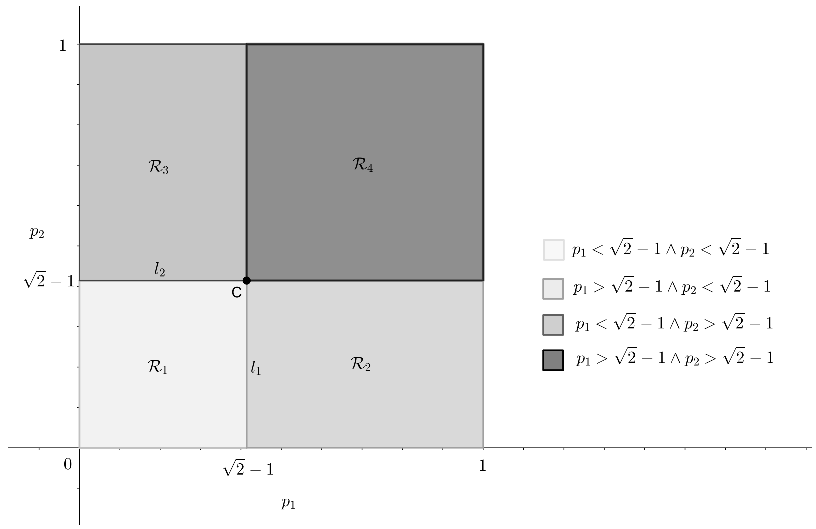


**Fig. 4.** Region, in the parameter space $(p_1, p_2)$, where the dynamics of two individual populations occurs.

## References


1.W. A. Coppel. The solution of equations by iteration. *Mathematical Proceedings of the Cambridge Philosophical Society*, 51(01):41–43, 1955.

2.S. Elaydi. *An Introduction to Difference Equations.* Springer, 3rd ed., 2005.

3.S. Elaydi. *Discrete Chaos: With Applications in Science and Engineering.* Chapman and Hall/CRC, 2nd ed., 2008.

4.S. Elaydi, R. Luís, and H. Oliveira. Local bifurcation in one-dimensional nonautonomous periodic difference equations. *International Journal of Bifurcation and Chaos*, 23(03):1350049(1–18), 2013.

5.M. I. Gomes F. Brilhante and D. Pestana. Panjer randomized Fibonacci model and dynamics instabilities in population growth models. *Chaotic Modeling and Simulations (CMSIM)*, 4:495–509, 2013.

6.W. Feller. *An Introduction to Probability Theory and its Applications*, vol. 1. John Wiley & Sons, 1957.

7.P. Kloeden. On Sharkovsky's cycle coexistence ordering. *Bulletin of the Australian Mathematical Society*, 20:171–177, 1979.

8.Y. A. Kuznetsov. *Elements of applied bifurcation theory.* Applied Mathematical Sciences 112. Springer, New York, 3rd ed., 2004.

9.R. Luís and H. Oliveira. Products of $2 \times 2$ matrices related to a non autonomous Fibonacci difference equation. *Applied Mathematics and Computation*, 226:101–116, 2014.

10.A. N. Sharkovsky. Coexistence of cycles of a continuous map of the line into itself (in russian). *Ukrainian Math. J.*, 16:61–71, 1964.

11.A. N. Sharkovsky. Coexistence of cycles of a continuous map of the line into itself (translation to English by J. Tolosa). *International Journal of Bifurcation and Chaos*, 0(5):1263–1273, 1995.

12.L. E. Sigler. *Fibonacci's Liber Abaci: Leonardo Pisano's Book of Calculation.* Sources and studies in the History of Mathematics and Physical Sciences. Springer, New York, 2002.