Fractal Geometry and Architecture Design: Case Study Review

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Abstract: The idea of buildings in harmony with nature can be traced back to ancient times. The increasing concerns on sustainability oriented buildings have added new challenges in building architecturereral design and called for new design responses. Sustainable design integrates and balances the human geometries and the natural ones. As the language of nature, it is, therefore, natural to assume that fractal geometry could play a role in developing new forms of aesthetics and sustainable architecturereral design. This paper gives a brief description of fractal geometry theory and presents its current status and recent developments through illustrative review of some fractal case studies in architecture design, which provides a bridge between fractal geometry and architecture design.
Keywords: Fractal geometry, Architecture design, Sustainability.

1. Introduction

The idea of buildings in harmony with nature can be traced back to ancient Egyptians, China, Greeks and Romans. At the beginning of 21st century, the increasing concerns on sustainability oriented on buildings have added new challenges in building architecture design and called for new design responses. As the language of nature [1,2], it is, therefore, natural to assume that fractal geometry could play a role in developing new forms of design of sustainable architecture and buildings.

Fractals are self-similar sets whose patterns are composed of smaller-scales copied of themselves, possessing self-similarity across scales. This means that they repeat the patterns to an infinitely small scale. A pattern with a higher

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fractal dimension is more complicated or irregular than the one with a lower dimension, and fills more space. In many practical applications, temporal and spatial analysis is needed to characterise and quantify the hidden order in complex patterns, fractal geometry is an appropriate tool for investigating such complexity over many scales for natural phenomena [2,3]. Order in irregular pattern is important in aesthetics as it embraces the concept of dynamic force, which shows a natural phenomenon rather than mechanical process. In architecture design terms, it represents design principle. Therefore, fractal geometry has played a significant role in architectureral design.

In spite of its growing applications, such works in literature are rather narrow, i.e. they mainly focus on applications for fractal design patterns on aesthetic considerations. Few works have related to a comprehensive and unified view of fractal geometry in structural design, for example, as it is intended in this study. We aim to fill in this gap by introducing fractals as new concepts and presenting its current status and recent developments in architecture through an illustrative review of some fractal case studies in design. The paper shows that incorporating the fractal way of thinking into the architecture design provides a language for an in-depth understanding of complex nature of architectural design in general. This study distils the fundamental properties and the most relevant characteristics of fractal geometry essential to architects and building scientists, initiates a dialogue and builds bridges between scientists and architects.

2. Basic Theory of Fractal Geometry

2.1. Basic Theory

The mathematical history of fractals began with mathematician Karl Weierstrass in 1872 who introduced a Weierstrass function which is continuous everywhere but differentiable nowhere [4]. In 1904 Helge von Koch refined the definition of the Weierstrass function and gave a more geometric definition of a similar function, which is now called the Koch snowflake [5], see Figure 1. In 1915, Waclaw Sielpinski constructed self-similar patterns and the functions that generate them. Georg Cantor also gave an example of a self-similar fractal [6]. In the late 19th and early 20th, fractals were put further by Henri Poincare, Felix Klein, Pierre Fatou and Gaston Julia. In 1975, Mandelbrot brought these work together and named it 'fractal'.

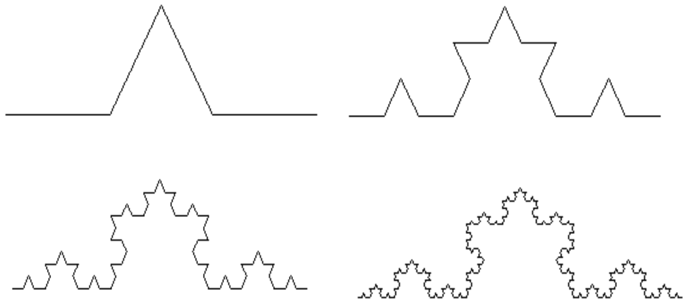

Fig. 1. Illustration of Koch Curve.

Fractals can be constructed through limits of iterative schemes involving generators of iterative functions on metric spaces [2]. Iterated Function System (IFS) is the most common, general and powerful mathematical tool that can be used to generate fractals. Moreover, IFS provides a connection between fractals and natural images [7,8]. It is also an important tool for investigating fractal sets. In the following, an introduction to some basic geometry of fractal sets will be approached from an IFS perspective. In a simple case, IFS acts on a segment to generate contracted copies of the segment which can be arranged in a plane based on certain rules. The iteration procedure must converge to get the fractal set. Therefore, the iterated functions are limited to strict contractions with the Banach fixed-point property.

Let (X, d) denotes a complete metric space and $H(X)$ the compact subsets of X, the Hausdorff distance is defined as

$$h(A,B) = \max\{d(A,B),\, d(B,A)\} \quad \forall A,\, B \in H(X) \qquad (1)$$

It is easy to prove that h is a metric on $H(X)$. Moreover, it can be proved that $(H(X), h)$ is also a complete metric space [7] which is called the space of fractals for X.

A contraction mapping, or contraction w: $X \;\alpha\; X$ has the property that there is some nonnegative real number $k \in [0,1)$, contraction factor k, such that

$$d(w(x), w(y)) \leq k\, d(x,y) \qquad (2)$$

In non-technical terms, a contraction mapping brings every two points closer in the metric space it maps. The Banach fixed point theorem guarantees the existence and uniqueness of fixed points of contract maps on metric spaces: If w: $X \;\alpha\; X$ is a contraction, then there exists one and only one $x \in X$ such that $w(x) = x$.

The Banach fixed point theorem has very important applications in many branches of mathematics. Therefore, generalisation of the above theorem has been extensively investigated, for example, in probabilistic metric spaces. The theorem also provides a constructive method to find fixed-point.

An IFS [9] is a set of contraction mappings w_i defined on (X, d) with contraction factors k_i for $i = 1,2,\ldots, N$. We denote it as $\{X; w_i, i = 1,2,\ldots, N\}$ and contraction factor $k = \max\{k_i, i = 1, 2,\ldots, N\}$. Hutchinson [9] proved an important theorem on a set of contraction mappings in which IFS is based: Let $\{X; w_i, i = 1,2,\ldots, N\}$ be an IFS with contraction factor k. Then W: $H(X) \to H(X)$ defined as

$$W(B) = \bigcup_{i=1}^{N} w_i(B) \quad \forall\ B \in H(X) \tag{3}$$

is a contraction mapping on $(H(X), h)$. From Banach's theorem, there exists a unique set $A \in H(X)$, the attractor of IFS, such that

$$A = W(A) = \bigcup_{i=1}^{N} w_i(A) \tag{4}$$

It can be seen that A is self-similar since it is expressed as a union of transformations (copies) of itself. The attractor A can be taken as a definition of deterministic fractals.

2.2. Fractal dimensions

Mandelbrot [2] proposed a simple but radical way to qualify fractal geometry through a fractal dimension based on a discussion of the length of the coast of England. The dimension is a statistical quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer scales. This definition is a simplification of the Hausdorff dimension that Mandelbrot used as a basis. We focus on this one and briefly mentions box-counting dimension because of its wide practical applications. However, it should be noted that there are many specific definitions of fractal dimensions, such as Hausdorff dimension, Rényi dimensions, box-counting dimension and correlation dimension, etc, and none of them should be treated as the universal one.

For $A \in H(X)$, let $n(A,\varepsilon)$, $\varepsilon<0$, denote the smallest number of closed balls of radius ε needed to cover A. If

$$D = \lim_{\varepsilon \to 0} \frac{\log n(A, \varepsilon)}{\log \frac{1}{\varepsilon}} \tag{5}$$

exists, then D is called the fractal dimension of A.

So $n(A,\varepsilon)$ is proportional to ε^{-D} as $\varepsilon \to 0$ or the exponent D is in $n(A,\varepsilon) = \varepsilon^{-D}$ which is the power law relationship. A power law describes a dynamic relationship between two objects which portrays a wide variety of natural and man-made phenomena. A key feature of the power law is that the power law relationship is independent of scales. A good example of intuition of fractal dimension is a line with the length of ε, where ε is the measuring length. Assume the line is divided in 3 equal parts and $\varepsilon = \frac{1}{3}$ then the simplified $n(A,\varepsilon) = \varepsilon^{-D}$ gives $3 = (1/3)^{-D}$ with $D = 1$. Similarly, the Koch curve's fractal dimension is $D = \frac{\log 4}{\log 3} = 1.26$.

Practically, the fractal dimension can only be used in the case where irregularities to be measured are in the continuous form. Natural objects offer a lot of variation which may not be self-similar. The Box-counting dimension is much more robust measure which is widely used even to measure images. To calculate the box-counting dimension, we need to place the image on a grid. The number of boxes, with size s_1, that cover the image is counted (n_1). Then the number of a smaller grid of boxes, with size s_2, is counted (n_2). The fractal dimension between two scales is then calculated by the relationship between the difference of the number of boxed occupied and the difference of inverse grid sizes [10]. In more chaotic and complex objects such as architecture and design, more flexible and robust measures, such as range analysis, midpoint displacement, etc, can be employed. For more detailed information, readers may refer to Bovill's book [10].

2.3. Examples of IFS applications

Fractal geometry is at the conceptual core of understanding nature's complexity and IFS provides an important concept for understanding the core design of the natural objects as well as approximating the natural design. In this subsection we outline the evolution of the idea of IFS with our calculation examples.

We know that the Banach's fixed-point theorem forms the basis of the IFS applications. However, applying the theorem in practiced raises two central questions. One is to find the attractor for a given IFS. The other is to find IFS for a given attractor, an inverse problem of the first.

For the first problem, the attractor can be obtained by successive approximations from any starting point theoretically. From a computational point of view, two techniques, deterministic and stochastic, can be applied. The deterministic algorithm starts with an arbitrary initial set to reach the attractor. The stochastic algorithm is often more complex but more efficient. A stochastic algorithm attributes to the IFS system with a set of probabilities by assigning a probability to each mapping, which is used to generate a random walk. If we start with any point and apply transformations iteratively, chosen according the probabilities attached, we will come arbitrarily close to the attractor. The associated probabilities determine the density of spatially contracted copies of the attractor. Therefore, the probabilities have no effect on the attractor but influence significantly the rendering of its approximations.

The second problem, the inverse problem, can be solved by Barnsley's Collage Theorem, a simple consequence of Banach's fixed point theorem. Such procedure was illustrated nicely through the 'Barnsley fern' in [9] and [11] using four-transformation IFS with associated probabilities. Figure 2 shows our calculation examples of fractals using four-transformation IFS with variations and their associated probabilities produced by Matlab, where 20000 iterations were set. These fractals actually have more than one attractor. In Figure 2, the four-transformation matrices are

$$A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0.18 & 0\\ 0 & 0 & 1\end{bmatrix}\ B=\begin{bmatrix}0.19 & -0.25 & 0\\ 0.22 & 0.23 & 1.7\\ 0 & 0 & 1\end{bmatrix}\ C=\begin{bmatrix}-0.15 & 0.28 & 0\\ 0.24 & 0.25 & 0.44\\ 0 & 0 & 1\end{bmatrix}\ D=\begin{bmatrix}0.76 & 0.04 & 0\\ -0.05 & 0.95 & 1.7\\ 0 & 0 & 1\end{bmatrix} \quad (6)$$

D has the probability 0.75 and others 0.083.

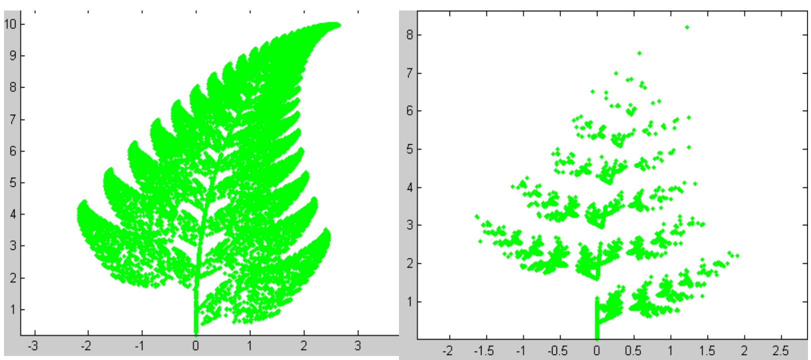

Fig. 2. Calculation examples of fractals using IFS with variations.

3. Applications of Fractal Geometry to Architecture Design

3.1. Applications of IFS

IFS provides wide range of architecture design applications in patterns and structures. Very often, IFS codes are used to generate fractals. For example, topology (layout) optimization has been proposed and is based on IFS representations with various applications [12]. Chang [13] proposed a hierarchical fixed point-searching algorithm for determining the original coordinates of a 2-D fractal set directly from its IFS code. The IFS code cane be modified according to the desired transformation. Figure 3 shows the Castle with different reflection directions generated by the modified IFS codes.

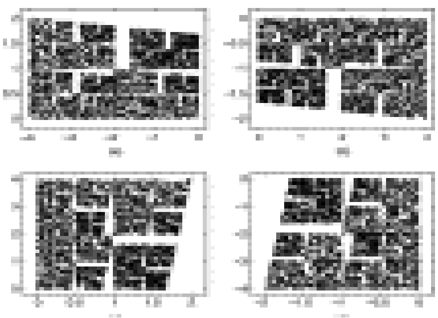

Fig. 3. Castle example generated by the modified IFS codes [13].

3.2. Applications of fractal geometry

Fractal geometry has been applied in architecture design widely to investigate fractal structures of cities [14] and successfully in building geometry [15,16] and design patterns [10].

Early fractal building patterns can be traced to ancient Maya settlements. Brown et al. analysed fractal structures of Maya settlements and found that fractals exhibit both within communities and across regions in various ways: at the intra-site, the regional levels and within archaeological sites. Moreover, spatial organisation in geometric patterns and order are also fractals, which presents in the size-frequency distribution, the rank-size relation among sites and the geographical clustering of sites [17]

In Europe, fractals were found in the early 12th century buildings. The floor of the cathedral of Anagni in Italy built in 1104 is adorned with dozens of mosaics in a form of a Sierpinski gasket fractal (See Figure 4).

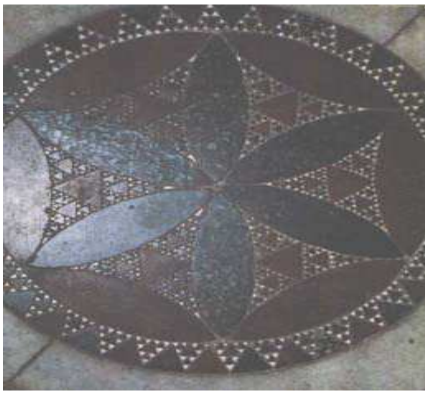

Fig. 4. The floor of the cathedral of Anagni in Italy [18].

Fractals have been applied to many elevation structures to exclusively address power and balance. Some very excellent examples of classical architecture can be seen in many parts of the Europe, in the Middle East and Asia which have effects of fractal elevations, for example, Reims' cathedral and Saint Paul church in France, Castel del Monte in Italy and many palaces in Venice (ca' Foscari, Ca' d'Oro, Duke Palace, Giustinian Palace) in Italy. Venice has been one of the most talked about fractals [18] (see Figure 5). More vital evidence shows that fractals exist in Gothic cathedrals in general. The pointed arch, an impression of elevation, appear in entrance, at windows and the costal arch with many scales and details [19]. Figure 6 displays the elevations of a five-floor tenement building in the historical part of Barcelona which shows self-organisation structure.

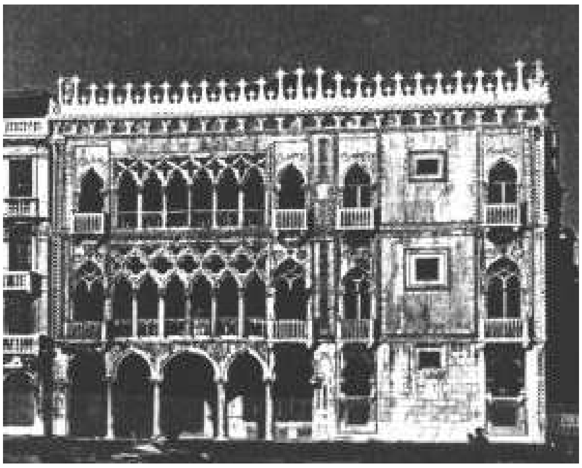

Fig. 5. Fractal building in Venice [18].

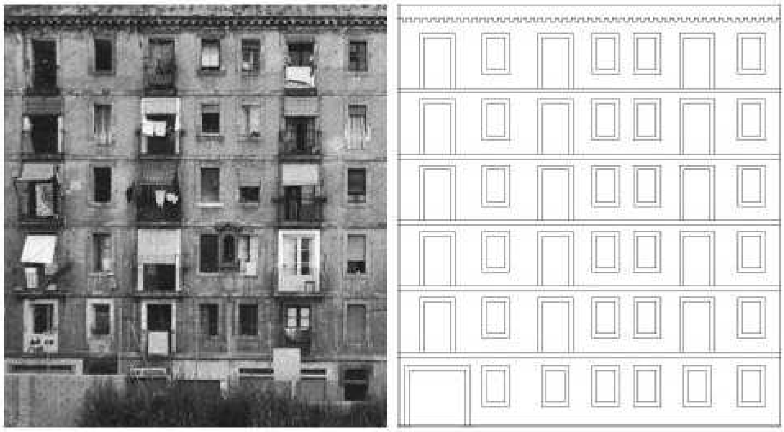

Fig. 6. A tenement house in the historical part of Barcelona, Spain: the elevation's photograph from the 90-s (left part); the geometric synthesis shows the original architecture design (middle part) [20].

In the Middle East, fractal patterns have been adopted widely in designing stucco, a typically Persian art form for the decoration of dome interiors. In Figure 7, the pattern in the dome interior has four attractors surrounding the main one at the center (Sarhangi).

Fig. 7. Stucco dome interior in a private house in Kashan [21].

In Asia, architectures with fractal structures have also been found in Humayun's Mausoleum, Shiva Shrine in India and the Sacred Stupa Pha That Luang in Laos. Fractals have been used to study Hindu temples. In China, some mosques in the west were more likely to incorporate such domes which are fractals. One important feature in Chinese architecture is its emphasis on symmetry which conveys a sense of grandeur [22].

Besides geographical localities, in recent times, the concept of fractals has been extended to many well known architectures including Frank Lloyd Wright's 'Robie House', 'Fallingwater', 'Palmer house' and 'Marion County Civic', which demonstrate that fractals have universal appeal and are visually satisfying because they are able to provide a sense of scale at different levels. Wright is probably the most representative of the organic architects. His designs grew out of the environment with regards to purpose, material and construction [10]. Fractals have inspired many great modern designers such as Zaha Hadid, Daniel Liebeskind, Frank Gehry and others with many notable fractal architectures [20]. Indeed, according to Ibrahim et al, architects and designers started to adopt fractals as a design form and tool in the 1980s [10]. Yessios et al. was among the first utilising fractals and fractal geometry design in architecture [23]. They developed a computer program to aid architecture using fractal generators. In 1990th, Durmisevic and Ciftcioglu applied fractal tree as an indicator of a road infrastructure in the architecture design and urban planning [24].

Wen et al. established the fractal dimension relations matrix table analysis to classify architecture design style patterns for the masterpieces of three modern architecture masters: Frank Lioyd Wright, Le Corbusier and Mies van der Rohe [25]. Figure 8 shows the results. It can be seen that the temporal trends of individuals vary. The fractal dimensions of Frank Lioyd Wright are average with low beginning in the early 1900th and end in the mid 1930th. The trend of Le Corbusier goes downside with gentle slope from mid 1900th to mid 1950th. For the period shown in the grape, the trend of Mies van der Rohe has the same trend as that of Frank Lioyd Wright from the early 1900th to the mid 1930th. The average trend of these three masters goes down in general starting from 1930th.

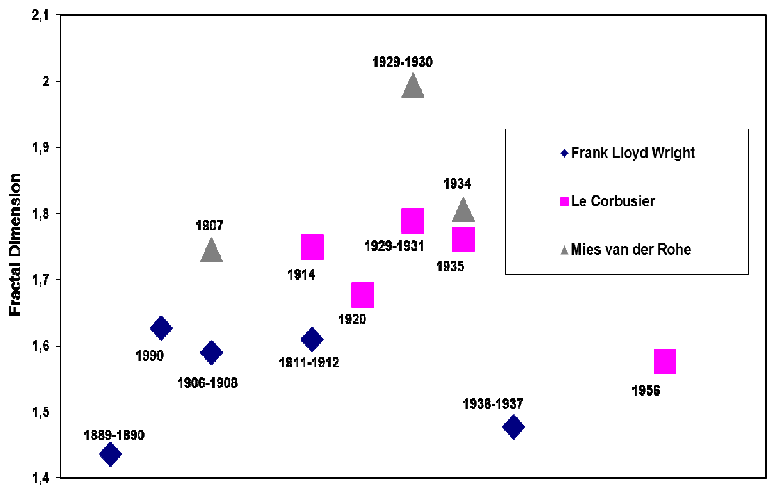

Fig. 8. Fractal dimensions for the masterpieces of three modern architects.

4. Conclusions

This paper has illustratively reviewed the fundamental concepts and properties of fractal geometry theory essential to architecture design, as well as the current state of its applications. Fractal geometry has important implications for buildings. The representative review shows that architecture design is not made to be isolated but to anticipate changes in the environment. Accumulation of technological modernisations, destroying, adapting and many changes have caused the design temporal and spatial diversity and complexity. More specifically, sustainable development in a building can be looked upon as adaptability and flexibility over time when it comes to responding to changing environments. Chaos and many other nonlinear theories have explained that extremely deterministic and linear processes are very fragile in maintaining stability over a wide range of conditions, whereas chaotic and fractal systems can function effectively over a wide range of different conditions, thereby offering adaptability and flexibility. In this context, fractal geometry theory offers an alternative for sustainable architectural design. This paper provides a bridge between building engineering and architecture and fractal geometry theory.

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