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Dynamical analysis of a fractional SIR model with treatment and quarantine

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Abstract. We propose a fractional SIR model with treatment and quarantine policies, whose dynamics is described by the Caputo fractional derivative. Local stability of the equilibrium points is studied, and the threshold value R_0 is found. Finally, some numerical simulations are presented for different values of the parameters.
Keywords: Fractional calculus, epidemiological models, numerical simulations.

1 Introduction

Traditionally, most of the mathematical models that describe the dynamics of direct transmission diseases are given by a system of differential equations. One of the concerns of epidemiological models is to use appropriate methods to study the occurrence of diseases. From this perspective, epidemiology currently uses mathematical modeling to describe the complex interactions between living beings. The most widely used models for describing transmissible diseases are those of the compartmental type, where each individual in a closed community is labeled by his state of health in relation to some illness. In this way, the individuals are located in compartments, each compartment representing the state of the development of the disease for every single individual. The most simple compartment model is composed of three compartments, labeled by the letters S, I and R. The compartment S includes all individuals who may become infected. When an individual of class S has contact with an infectious individual and becomes itself infected, he moves to class I. After the infectivity period, the individual will belong to the class of the Recovered R, staying for some time or permanently immune to the disease. More complex models can be used, for example, one can include a class of individuals that possess natural immunity to the disease or are infected, but can not spread it yet. In 1927, Kermack and McKendrick proposed the classical SIR model [14], which played a major influence on the development of mathematical models for disease spread. According to this model, all individuals are equally susceptible to the disease, and a complete immunity is conferred after the infection. Since then, numerous works have come to light, as the model assumptions are modified. For example,

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we can consider the SIR model [4,20,23], the SIS model [11,25], the SIRS model [15,19], the SEIR model [16,17], the MSEIR model [13], among others.

More recently, with the advancement of fractional calculus, a new approach is appearing by replacing ordinary derivatives with fractional derivatives. There are two important reasons to consider fractional derivatives in epidemiological models. First, fractional derivatives are nonlocal operators and may be more suitable for long-time behaviour studies. Secondly, by considering the order of the derivative an arbitrary real $\alpha > 0$, not necessarily an integer number, we can model more efficiently real data to the theoretical model. To mention a few works on epidemiological fractional models, we refer the reader to [1,5–8,10,21,24].

For the reader's convenience, we start with a short exposition on fractional calculus [12,22]. Let $\alpha > 0$ be a real, $n = [\alpha] + 1$ an integer, and $x : [a, b] \to \mathbb{R}$ an integrable function. The Riemann–Liouvile fractional integral of x of order α is given by the expression

$$I_{a+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \quad t > a,$$

and the Riemann–Liouville fractional derivative of x of order α is defined as

$$D_{a+}^{\alpha} x(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{n-\alpha} x(t), \quad t > a.$$

Another important concept is the Caputo fractional derivative, given by

$${}^{C}D_{a+}^{\alpha} x(t) = D_{a+}^{\alpha} \left[x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!} (t-a)^k \right], \quad t > a.$$

From the definition it is clear that the Riemann–Liouville fractional derivative of a constant function is not zero, while ${}^{C}D_{a+}^{\alpha} K = 0$, where K is a real number. It is worth to mention that, if $\alpha \in \mathbb{N}$, then ${}^{C}D_{a+}^{\alpha} x(t) = x^{(\alpha)}(t)$, and if x is of class C^n, then

$${}^{C}D_{a+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \quad t > a.$$

Thus, in opposite to integer-order calculus, fractional derivatives are non-local operators and, thus, contain memory.

The outline of the paper is the following. In Section 2, we present our proposed model, given by a system of fractional differential equations. Then, in Section 3, we prove the existence of nonnegative solutions to the system. The equilibrium points and basic reproduction number R_0 are considered in Section 4. We end by presenting some examples in final Section 5, with numerical simulations.

2 Model formulation

In this work, we begin with a SIR model (Susceptible, Infected and Recovered), adding, however, more classes. Once infected, some patients will receive

treatment and move to a new class, which we denominate by H compartment. Others infected wont receive any treatment and will be quarantined during the period of infection, labeled as Q class. The model that we propose in our work is described by a system of fractional differential equations (1) of order $\alpha \in (0, 1)$:

$$\begin{cases} {}^{C}D^{\alpha}_{0+}s(t) &= \mu^{\alpha} - \beta^{\alpha} i(t)s(t) - \mu^{\alpha} s(t) \\ {}^{C}D^{\alpha}_{0+}i(t) &= \beta^{\alpha} i(t)s(t) - (\epsilon^{\alpha} + \mu^{\alpha})i(t) \\ {}^{C}D^{\alpha}_{0+}h(t) &= u\epsilon^{\alpha} i(t) - (\tau^{\alpha} + \mu^{\alpha})h(t) \\ {}^{C}D^{\alpha}_{0+}q(t) &= (1-u)\epsilon^{\alpha} i(t) - (\gamma^{\alpha} + \mu^{\alpha})q(t) \\ {}^{C}D^{\alpha}_{0+}r(t) &= \tau^{\alpha} h(t) + \gamma^{\alpha} q(t) - \mu^{\alpha} r(t), \end{cases} \tag{1}$$

with the initial conditions

$$s(0) = s_0,\ i(0) = i_0,\ h(0) = h_0,\ q(0) = q_0,\ r(0) = r_0, \tag{2}$$

with $s_0, i_0, h_0, q_0, r_0 \in \mathbb{R}^{+}_{0}$. The variables s, i, h, q and r represent the fraction of the total population in each of the five categories, $s = S/N$, $i = I/N$, $h = H/N$, $q = Q/N$ and $r = R/N$, where N denotes the population size. Since ${}^{C}D^{\alpha}_{0+}N(t) = 0$, the population size is constant along time. We assume that the fractional derivatives ${}^{C}D^{\alpha}_{0+}s$, ${}^{C}D^{\alpha}_{0+}i$, ${}^{C}D^{\alpha}_{0+}h$, ${}^{C}D^{\alpha}_{0+}q$, and ${}^{C}D^{\alpha}_{0+}r$ exist and are continuous at every point $t \geq 0$ (for example, it is enough to assume that functions s, i, h, q, and r are of class C^1). The parameters of the model are displayed in Table 1, and the disease transmission dynamics is displayed in Figure 1.

β	disease transmission rate
ϵ	time it takes to decide who will receive treatment
τ	rate of recovery from the disease due to treatment
γ	natural rate of recovery from the disease
u	percentage of those infected who will receive treatment

Table 1: Parameters of the model.

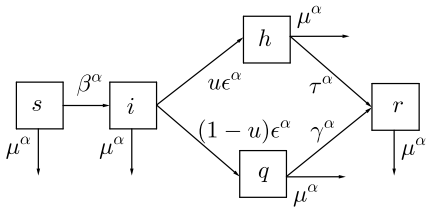

Fig. 1: Disease transmission dynamics.

On one hand, we are interested in reducing the costs of treatment, that is, the parameter u. At the same time, we want to prevent a large part of the population from being quarantined and infected for too long. These issues may be approached

as an optimal control problem. We intend to find the optimal value u that minimizes the functional

$$J(u) = \omega u + \int_0^T q(t)\,dt,$$

subject to the system (1).

3 Existence of non-negative solutions

In this section we prove the existence and uniqueness of the solution for system (1). Also, we prove that the solution of the system is non-negative.

Theorem 1. *There exists a unique solution for problem* (1).

Proof. The existence of solution follows immediately from [18, Theorem 3.1] and the uniqueness from [18, Remark 3.2].

Next, we prove the non-negativity of solution. First, we need the following two lemmas.

Lemma 1. *[10] Let x be a function such that x and ${}^C D^{\alpha}_{a+}x$ are continuous, for $\alpha \in (0,1]$. Then, for all $t \in (a,b]$, there exists some $c \in (a,t)$ satisfying the condition*

$$x(t) = x(a) + \frac{1}{\Gamma(\alpha+1)}\,{}^C D^{\alpha}_{a+}x(c)(t-a)^{\alpha}.$$

Lemma 2. *[3] Given a real $\alpha \in (0,1)$, an integer $m \in \mathbb{N}$, and consider the vectors $X = (x_1,\ldots,x_m)$ and $Y = (y_1,\ldots,y_m)$. For each $i = 1,\ldots,m$, let $f_i : [a,b]\times\mathbb{R}^m \to \mathbb{R}$ be a continuous function and Lipschtiz with respect to the second component, that is,*

$$|f_i(t,X) - f_i(t,Y)| \le L_i\|X - Y\|.$$

Let us denote $f = (f_1,\ldots,f_m)$, and consider the two fractional differential equations

$${}^C D^{\alpha}_{a+}X(t) = f(t,X) + \frac{1}{k} \quad \text{and} \quad {}^C D^{\alpha}_{a+}X(t) = f(t,X), \tag{3}$$

with the same initial conditions, where k is a positive integer. If ${}_kX^{\star} = ({}_kx_1^*,\ldots,{}_kx_m^*)$ and $X^{\star} = (x_1^*,\ldots,x_m^*)$ are the solutions of (3)*, respectively, then ${}_kX^{\star}(t) \to X^{\star}(t)$ as k goes to infinity, for all $t \in [a,b]$.*

Theorem 2. *The solution of system* (1) *is non-negative.*

Proof. Consider the following system with an extra parameter $k \in \mathbb{N}$:

$$\begin{cases} {}^C D^{\alpha}_{0+}s(t) = \mu^{\alpha} - \beta^{\alpha} i(t)s(t) - \mu^{\alpha}s(t) + 1/k \\ {}^C D^{\alpha}_{0+}i(t) = \beta^{\alpha} i(t)s(t) - (\epsilon^{\alpha} + \mu^{\alpha})i(t) + 1/k \\ {}^C D^{\alpha}_{0+}h(t) = u\epsilon^{\alpha} i(t) - (\tau^{\alpha} + \mu^{\alpha})h(t) + 1/k \\ {}^C D^{\alpha}_{0+}q(t) = (1-u)\epsilon^{\alpha} i(t) - (\gamma^{\alpha} + \mu^{\alpha})q(t) + 1/k \\ {}^C D^{\alpha}_{0+}r(t) = \tau^{\alpha} h(t) + \gamma^{\alpha} q(t) - \mu^{\alpha} r(t) + 1/k. \end{cases} \tag{4}$$

First, we prove that solution $(s_k^{\star}(t), i_k^{\star}(t), h_k^{\star}(t), q_k^{\star}(t), r_k^{\star}(t))$ of (4), with the initial conditions (2), is non-negative. Suppose, by *reductio ad absurdum*, that one of the functions is negative at some point $t > 0$, and let

$$t_0 = \inf\{t > 0 \,|\, (s_k^{\star}(t), i_k^{\star}(t), h_k^{\star}(t), q_k^{\star}(t), r_k^{\star}(t)) \notin (R_0^+)^5\}.$$

Under this hypothesis, at $t = t_0$, we have that

$$(s_k^\star(t_0), i_k^\star(t_0), h_k^\star(t_0), q_k^\star(t_0), r_k^\star(t_0)) \in (R_0^+)^5$$

and one of the quantities $s_k^\star(t_0)$, $i_k^\star(t_0)$, $h_k^\star(t_0)$, $q_k^\star(t_0)$, or $r_k^\star(t_0)$, is equal to zero. Without loss of generality, suppose that $s_k^\star(t_0) = 0$. By (4), we obtain that

$$^C D_{0+}^\alpha s_k^\star(t_0) = \mu^\alpha + \frac{1}{k} > 0,$$

and by continuity of $^C D_{0+}^\alpha s_k^\star$, we prove that $^C D_{0+}^\alpha s_k^\star(t) > 0$ at some interval $[t_0, t_0+\zeta)$, for some $\zeta > 0$. Applying Lemma 1, we conclude that $s_k^\star$ is strictly increasing on the interval $[t_0, t_0 + \zeta)$, and so it is positive. In conclusion, $s_k^\star$ is non-negative. Since

$$^C D_{0+}^\alpha i|_{i=0} = 1/k > 0, \quad ^C D_{0+}^\alpha h|_{h=0} = u\epsilon^\alpha i(t) + 1/k > 0,$$

$$^C D_{0+}^\alpha q|_{q=0} = (1-u)\epsilon^\alpha i(t) + 1/k > 0, \quad ^C D_{0+}^\alpha r|_{r=0} = \tau^\alpha h(t) + \gamma^\alpha q(t) + 1/k > 0,$$

using the same argument as before, we conclude that all the functions are non-negative. By Lemma 2, letting $k \to \infty$, we prove that the solution of (1)-(2) is non-negative, which ends the proof.

4 Equilibrium points and basic reproduction number

In this section, we study the equilibrium points for (1). They are found by solving the system

$$^C D_{0+}^\alpha s(t) = 0; \quad ^C D_{0+}^\alpha i(t) = 0; \quad ^C D_{0+}^\alpha h(t) = 0; \quad ^C D_{0+}^\alpha q(t) = 0; \quad ^C D_{0+}^\alpha r(t) = 0. \tag{5}$$

It is easy to verity that $P_F = (1, 0, 0, 0, 0)$ is a solution of (5), and it is called a disease free equilibrium point. To study its stability, we consider the Jacobian matrix of system (1):

$$J(P) = \begin{bmatrix} -\beta^\alpha i - \mu^\alpha & -\beta^\alpha s & 0 & 0 & 0 \\ \beta^\alpha i & \beta^\alpha s - \epsilon^\alpha - \mu^\alpha & 0 & 0 & 0 \\ 0 & u\epsilon^\alpha & -\tau^\alpha - \mu^\alpha & 0 & 0 \\ 0 & (1-u)\epsilon^\alpha & 0 & -\gamma^\alpha - \mu^\alpha & 0 \\ 0 & 0 & \tau^\alpha & \gamma^\alpha & -\mu^\alpha \end{bmatrix}.$$

Evaluated at the disease free equilibrium point,

$$J(P_F) = \begin{bmatrix} -\mu^\alpha & -\beta^\alpha & 0 & 0 & 0 \\ 0 & \beta^\alpha - \epsilon^\alpha - \mu^\alpha & 0 & 0 & 0 \\ 0 & u\epsilon^\alpha & -\tau^\alpha - \mu^\alpha & 0 & 0 \\ 0 & (1-u)\epsilon^\alpha & 0 & -\gamma^\alpha - \mu^\alpha & 0 \\ 0 & 0 & \tau^\alpha & \gamma^\alpha & -\mu^\alpha \end{bmatrix}. \tag{6}$$

The spectrum of matrix (6) is

$$\sigma(J(P_F)) = \{-\tau^\alpha - \mu^\alpha, -\gamma^\alpha - \mu^\alpha, -\mu^\alpha, \beta^\alpha - \epsilon^\alpha - \mu^\alpha\}.$$

Therefore, the disease free equilibrium point P_F is locally asymptotically stable if

$$\frac{\beta^\alpha}{\epsilon^\alpha + \mu^\alpha} < 1.$$

In fact, an equilibrium point is locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix verify the following condition [2]:

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}.$$

The number

$$R_0 = \frac{\beta^\alpha}{\epsilon^\alpha + \mu^\alpha}$$

is known as basic reproduction number.

One other solution of (5), called endemic equilibrium point, is $P_E = (s^\star, i^\star, h^\star, q^\star, r^\star)$, with

$$s^\star = \frac{1}{R_0}, \quad i^\star = \frac{\mu^\alpha}{\beta^\alpha}(R_0 - 1), \quad h^\star = \frac{u\epsilon^\alpha}{\tau^\alpha + \mu^\alpha}\frac{\mu^\alpha}{\beta^\alpha}(R_0 - 1),$$
$$q^\star = \frac{(1-u)\epsilon^\alpha}{\gamma^\alpha + \mu^\alpha}\frac{\mu^\alpha}{\beta^\alpha}(R_0 - 1), \quad r^\star = \frac{(R_0 - 1)\epsilon^\alpha}{\beta^\alpha}\left(\frac{u\tau^\alpha}{\tau^\alpha + \mu^\alpha} + \frac{(1-u)\gamma^\alpha}{\gamma^\alpha + \mu^\alpha}\right),$$

if $s^\star, i^\star, h^\star, q^\star, r^\star$ are between 0 and 1. With respect to this equilibrium point, the spectrum of the Jacobian matrix is

$$\sigma(J(P_E)) = \{-\tau^\alpha - \mu^\alpha, -\gamma^\alpha - \mu^\alpha, -\mu^\alpha, \lambda_E\},$$

where

$$\lambda_E = \frac{-\mu^\alpha R_0 \pm \sqrt{(\mu^\alpha R_0)^2 - 4\beta^\alpha\mu^\alpha(R_0 - 1)/R_0}}{2}.$$

Thus, the endemic free equilibrium point P_E is locally asymptotically stable if

$$|\arg(\lambda_E)| > \alpha\frac{\pi}{2}.$$

5 Numerical simulations

The method that we will use to solve numerically the optimal control problem consists in replacing the Caputo fractional derivative by the Grünwald–Letnikov fractional derivative. The procedure is explained next. First, consider a partition of the interval $[0, T]$ given by the sequence of points $t_j = jT/N$, for $j = 0, 1, ..., N$. Then, we use the approximation

$${}^C D^\alpha_{0+} x(t_j) \approx \frac{1}{h^\alpha}\sum_{k=0}^{j} w^\alpha_k x(t_{j-k}) - \frac{x(0)}{\Gamma(1-\alpha)}(t_j)^{-\alpha} =: {}^C \tilde{D}^\alpha_{0+} x(t_j),$$

where w^α_k represent the generalization of binomial coefficients to real numbers:

$$w^\alpha_k = (-1)^k \binom{\alpha}{k}.$$

With this, we rewrite the optimal control problem in its discrete form: minimize the sum

$$\omega u + \frac{T}{N}\sum_{k=1}^{N} q(t_k),$$

subject to the system

$$\begin{cases} {}^{C}\tilde{D}^{\alpha}_{0+}s(t_j) &= \mu^{\alpha} - \beta^{\alpha} i(t_j)s(t_j) - \mu^{\alpha} s(t_j) \\ {}^{C}\tilde{D}^{\alpha}_{0+}i(t_j) &= \beta^{\alpha} i(t_j)s(t_j) - (\epsilon^{\alpha} + \mu^{\alpha})i(t_j) \\ {}^{C}\tilde{D}^{\alpha}_{0+}h(t_j) &= u\epsilon^{\alpha} i(t_j) - (\tau^{\alpha} + \mu^{\alpha})h(t_j) \\ {}^{C}\tilde{D}^{\alpha}_{0+}q(t_j) &= (1-u)\epsilon^{\alpha} i(t_j) - (\gamma^{\alpha} + \mu^{\alpha})q(t_j) \\ {}^{C}\tilde{D}^{\alpha}_{0+}r(t_j) &= \tau^{\alpha} h(t_j) + \gamma^{\alpha} q(t_j) - \mu^{\alpha} r(t_j), \end{cases} \tag{7}$$

and to the initial conditions

$$s(0) = s_0,\ i(0) = i_0,\ h(0) = h_0,\ q(0) = q_0,\ r(0) = r_0, \tag{8}$$

for $j \in \{1, \ldots, N\}$. With respect to the parameters, the unit of time is a week. For the birth and death rates, we fix $\mu = 0.0002$. The transmission rate is $\beta = 1$, and in one day the decision if an infected will receive or not treatment is taken so that $\epsilon = 7$. With treatment, the time needed to recover from the disease is 2 days, so $\tau = 7/2$, and without treatment, we need two weeks ($\gamma = 1/2$). We test our model with and without control, fixing different values for u and ω. First, we solve system (7)-(8) and consider the values $u \in \{0.1, 0.9\}$. The results when solving (7)-(8) are displayed in Figures 2 and 3.

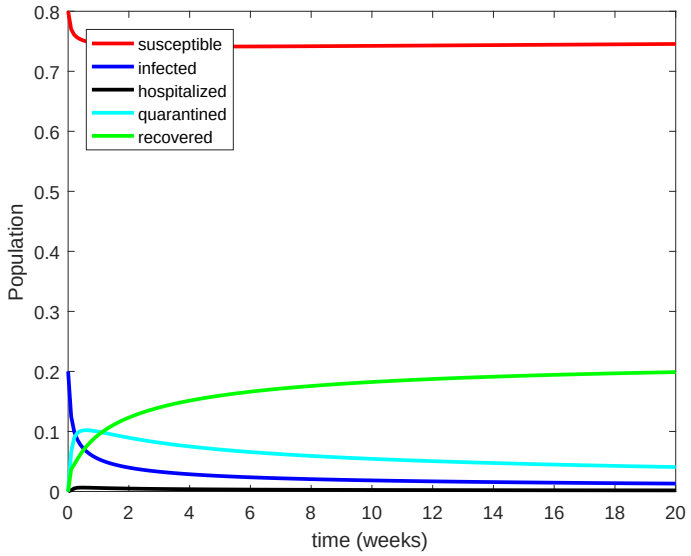

Fig. 2: Results obtained for $u = 0.1$.

As expected, for smaller values of u, the number of infected going to quarantine is higher. Now, we look again to system (7)-(8) from an optimal control point of view, by also including the cost functional:

$$\omega u + \frac{T}{N} \sum_{k=1}^{N} q(t_k) \to \min .$$

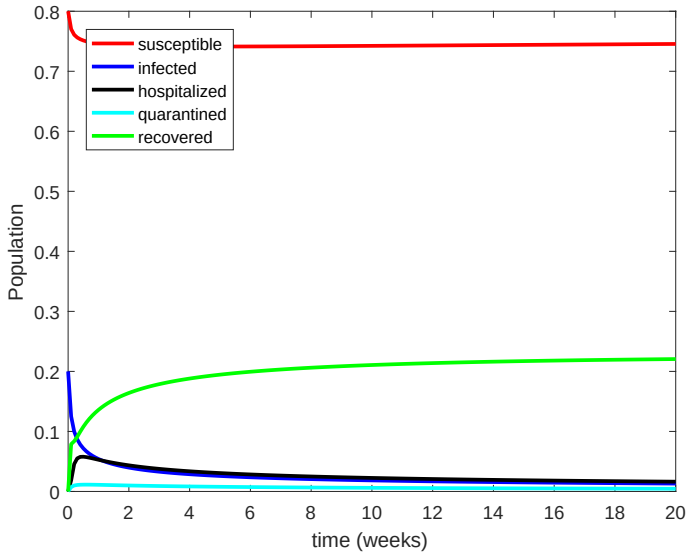

Fig. 3: Results obtained for $u = 0.9$.

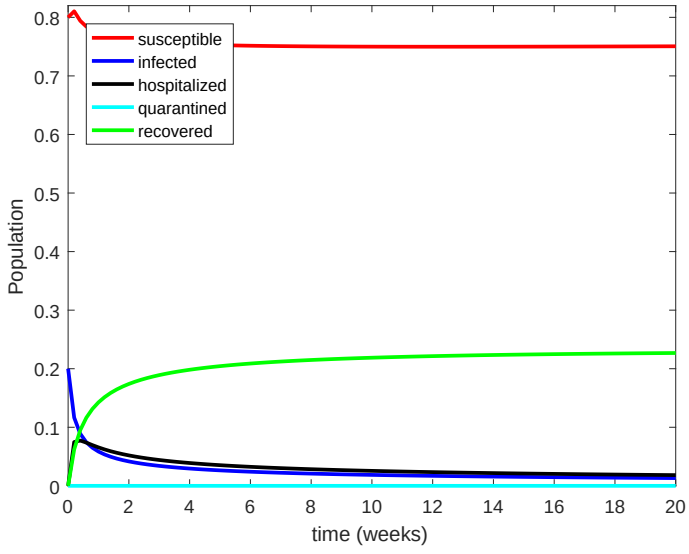

Fig. 4: Results obtained for $\omega = 1$.

With respect to parameter ω, we consider $\omega \in \{1, 10\}$, and the results are displayed in Figures 4 and 5. Increased the value of ω, the system will force the control u to decrease, and this will produce an increase on the number of quarantined individuals.

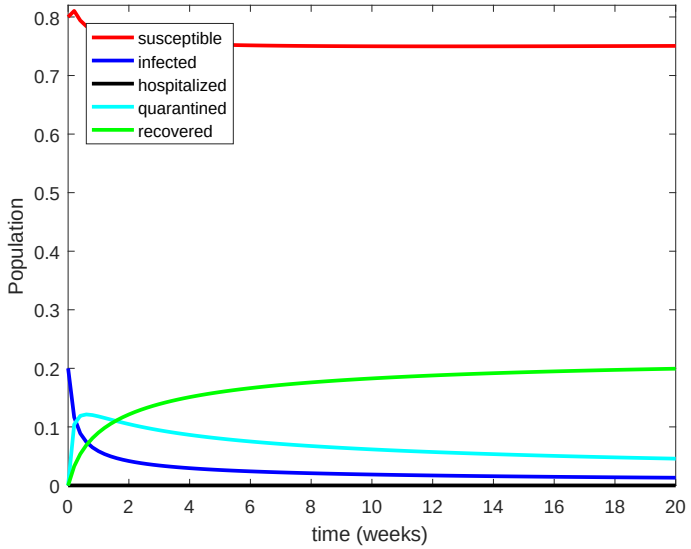

Fig. 5: Results obtained for $\omega = 10$.

Acknowledgments

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