

The Fluid Dynamics of Spin - a Fisher Information Perspective and Comoving Scalars

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Abstract. Two state systems have wide applicability in quantum modelling of various systems. Here I return to the original two state system, that is Pauli's electron with a spin. And show how this system can be interpreted as a vortical fluid. The similarities and difference between spin flows and classical ideal flows are elucidated. It will be shown how the internal energy of the spin fluid can partially be interpreted in terms of Fisher Information.

Keywords: Spin, Fluid dynamics.

1 Introduction

The Copenhagen interpretation of quantum mechanics is probably the most prevalent approach. This approach defies any ontology quantum theory and declares it to be completely epistemological in accordance to the Kantian [1] conception of reality. However, in addition to this approach we see the development of another school that believed in the realism of the wave function. This approach that was championed by Einstein and Bohm [2–4] led to other interpretations of quantum mechanics among them the fluid interpretation due to de Madlung [5,6] which interpreted the modulus square as the fluid density and the phase as a potential of a velocity field. However, this model was limited to spin less electrons and could not take into account a complete set of electron attributes even for non relativistic electrons.

A spin dependent non relativistic quantum equation was first introduced by Wolfgang Pauli in 1927 [7]. This equation contained a Hamiltonian which is a two dimensional operator matrix. Such two dimensional operator Hamiltonians were later found useful for many systems that required quantum modelling among them molecules and solids. Such two dimensional operator matrix Hamiltonians are abundant in the literature ([8] - [21]). The question now arises whether such a theory admits a fluid dynamical interpretation. This question seems of paramount importance as the proponents of the non-realistic Copenhagen interpretation of quantum mechanics usually use the concept of spin as a proof that some elements of nature are inherently quantum and have no classical analogue or interpretation. A Bohmian interpretation of the Pauli equation



was given by Holland and others [3], however, the relation of this equation to fluid dynamics and the concept of spin vorticity were not introduced. This situation was amended in a recent paper describing spin fluid dynamics [22].

The formulation of Pauli's theory in terms of a fluid theory leads us directly to the nineteenth century work of Clebsch [23,24] and the variational formulation of fluid dynamics. Variational principles for non-magnetic barotropic fluid dynamics are well known. A four function variational formulation of Eulerian barotropic fluid dynamics was derived by Clebsch [23,24] and later by Davidov [25] who's main motivation was to quantize fluid dynamics. Since the work was written in Russian, it was unknown in the west. Lagrangian fluid dynamics (as opposed to Eulerian fluid dynamics) was formulated through a variational principle by Eckart [26]. Initial western attempts to formulate Eulerian fluid dynamics in terms of a variational principle, were described by Herivel [27], Serrin [28] and Lin [29]. However, the variational principles developed by the above authors were very cumbersome containing quite a few "Lagrange multipliers" and "potentials". The range of the total number of independent functions in the above formulations ranges from eleven to seven which exceeds by many the four functions appearing in the Eulerian and continuity equations of a barotropic flow. And therefore did not have any practical use or applications. Seliger & Whitham [30] have developed a variational formalism depending on only four variables for barotropic flow and thus repeated the work of Davidov's [25] which they were unaware of. Lynden-Bell & Katz [31] have described a variational principle in terms of two functions the load λ (to be described below) and density ρ . However, their formalism contains an implicit definition for the velocity \mathbf{v} such that one is required to solve a partial differential equation in order to obtain both \mathbf{v} in terms of ρ and λ as well as its variations. Much the same criticism holds for their general variational for non-barotropic flows [32]. Yahalom & Lynden-Bell [33] overcame this limitation by paying the price of adding an additional single function. Their formalism allowed arbitrary variations and the definition of \mathbf{v} is explicit.

A fundamental problem in the fluid mechanical interpretation of quantum mechanics still exist. This refers to the meaning of thermodynamic quantities which are part of fluid mechanics. In thermodynamics Concepts like specific enthalpy, pressure and temperature are derivatives of the specific internal energy which is given in terms of the equation of state as function of entropy and density. The internal energy is a part of any Lagrangian density attempting to describe fluid dynamics. The form of the internal energy can in principle be explained on the basis of the microscopic composition of the fluid, that is the atoms and molecules from which the fluid is composed and their interactions using statistical mechanics. However, the quantum fluid has no microscopic structure and yet analysis of the equations of both the spin less [5,6] and spin [22] quantum fluid dynamics shows that terms analogue to internal energies appear in both cases. The question then arises where do those internal energies come from, surely one would not suggest that the quantum fluid has a microscopic sub structure as this will defy the conception of the electron as a fundamental particle. The answer to this question seems to come from an entirely different discipline of measurement theory [38,41]. Fisher information

a basic notion of measurement theory is a measure of the quality of the measurement of any quantity. It was shown [41] that this concept is proportional to the internal energy of a spin less electron and can explain most parts of the internal energy of an electron with spin. An attempt to unify most physical theories using Fisher information is described in a book by Frieden [40]. It is conjectured that there exist a velocity field such that the Fisher information will give a complete explanation for the spin fluid internal energy.

We will begin this paper by introducing Fisher information and the concept of probability amplitude. This will be followed by a discussion of Schrödinger equation and its interpretation in terms of Madelung fluid dynamics. The variational principle of Madelung fluid dynamics will be described and its relations to Fisher information. Then we introduce Pauli's equation and interpret it in terms Clebsch variables and discuss the similarities and differences between Clebsch and spin fluid dynamics and their variational principles. The concept of Fisher information will be introduced in the frame work of spin fluid dynamics. Finally we discuss the modifications that are needed in the velocity field to achieve comoving natural spin labels.

2 Fisher Information

Let there be a random variable X with probability density function (PDF) $f_X(x)$. The Fisher Information for a PDF which is translationally invariant is given by the form:

$$F_I = \int dx \left(\frac{df_X}{dx} \right)^2 \frac{1}{f_X} \quad (1)$$

It was shown [38,40] that the standard deviation σ_X of any random variable is bounded from below such that:

$$\sigma_X \geq \sigma_{Xmin} = \frac{1}{F_I} \quad (2)$$

Hence the higher the Fisher information we have about the variable the smaller standard deviation we may achieve and thus our knowledge about the value of this random variable is greater. This is known as the Cramer Rao inequality. Fisher information is most elegantly introduced in terms of the probability amplitude:

$$f_X = a^2 \Rightarrow F_I = 4 \int dx \left(\frac{da}{dx} \right)^2 \quad (3)$$

In this work we will be interested in a three dimensional random variable designating the position of an electron, hence:

$$F_I = \int d^3x (\nabla f_X)^2 \frac{1}{f_X} = 4 \int d^3x (\nabla a)^2 \equiv \int d^3x \mathcal{F}_I \quad (4)$$

In the above $\mathcal{F}_I \equiv 4(\nabla a)^2$ is the Fisher information density.

3 Schrödinger's Theory Formulated in Terms of Fluid Mechanics

3.1 Background

The modulus-phase formulation of a single particle quantum mechanics, leads very directly to the equation of continuity and to the Bernoulli (Hamilton-Jacobi) equation. These equations have formed the basic building blocks in Bohm's formulation of non-relativistic Quantum Mechanics [2].

The earliest appearance of the non-relativistic continuity equation is due to Schrödinger himself [37], obtained from his time-dependent wave-equation:

$$i\hbar\dot{\psi} = \hat{H}\psi, \quad \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V \quad (5)$$

in the above $i = \sqrt{-1}$ and ψ is the complex wave function. $\dot{\psi} = \frac{\partial\psi}{\partial t}$ is the partial time derivative of the wave function. $\hbar = \frac{h}{2\pi}$ is Planck's constant divided by 2π and m is the particles mass, V is the potential of a force acting on the particle. The Lagrangian density \mathcal{L} for the non-relativistic electron is written as:

$$\mathcal{L} = -\frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi - V\psi^*\psi + \frac{1}{2}i\hbar(\psi^*\dot{\psi} - \dot{\psi}^*\psi) \quad (6)$$

If now the modulus a and phase ϕ are introduced through:

$$\psi = ae^{i\phi} \quad (7)$$

the Lagrangian density takes the form:

$$\mathcal{L} = -\frac{\hbar^2}{2m}[(\nabla a)^2 + a^2(\nabla\phi)^2] - a^2V - \hbar a^2\frac{\partial\phi}{\partial t} \quad (8)$$

The variational derivative of this with respect to ϕ yields the continuity equation :

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \rightarrow \frac{\partial\hat{\rho}}{\partial t} + \nabla \cdot (\hat{\rho}\hat{\mathbf{v}}) = 0 \quad (9)$$

in which the mass density is defined as: $\hat{\rho} = ma^2$ and the velocity is $\hat{\mathbf{v}} = \frac{\hbar}{m}\nabla\phi$.

Variationally deriving with respect to a leads to the Hamilton-Jacobi equation :

$$\frac{\delta\mathcal{L}}{\delta a} = 0 \rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S^2 + V = \frac{\hbar^2\nabla^2 a}{2ma} \quad (10)$$

in which: $S = \hbar\phi$. The right hand side of the above equation contains the "quantum correction". These results are elementary, but their derivation illustrates the advantages of using the two variables, phase and modulus, to obtain equations of motion that have a substantially different form than the familiar Schrödinger equation (although having the same mathematical content) and have straightforward physical interpretations [2]. The interpretation is, of course, connected to the modulus being a physical observable (by Born's interpretational postulate) and to the phase having a similar though somewhat

more problematic status. (The "observability" of the phase has been discussed in the literature by various sources, e.g. in [39] and, in connection with a recent development, in [10,12].)

Another possibility to represent the quantum mechanical Lagrangian density is using the logarithm of the amplitude $\lambda = \ln a$, $a = e^\lambda$. In that particular representation the Lagrangian density takes the following symmetrical form:

$$\mathcal{L} = -e^{2\lambda} \left\{ \frac{\hbar^2}{2m} [(\nabla\lambda)^2 + (\nabla\phi)^2] + \hbar \frac{\partial\phi}{\partial t} + V \right\} \quad (11)$$

3.2 Similarities Between Potential Fluid Dynamics and Quantum Mechanics

In writing the Lagrangian density of quantum mechanics in the modulus-phase representation, equation (8), one notices a striking similarity between this Lagrangian density and that of potential fluid dynamics (fluid dynamics without vorticity) as represented in the work of Clebsch [30]. The connection between fluid dynamics and quantum mechanics of an electron was already discussed by Madelung [5] and in Holland's book [3]. However, the discussion by Madelung refers to the equations only and does not address the variational formalism which we discuss here.

If a flow satisfies the condition of zero vorticity, i.e. the velocity field \mathbf{v} is such that $\nabla \times \mathbf{v} = 0$, then there exists a function ν such that $\mathbf{v} = \nabla\nu$. The above statement is equivalent to taking a Clebsch representation of the velocity field but with $\alpha = \beta = 0$. In that case one can describe the fluid mechanical system with the following Lagrangian density:

$$\hat{\mathcal{L}} = - \left[\frac{\partial\nu}{\partial t} + \frac{1}{2} (\nabla\nu)^2 + \varepsilon(\rho) + \mathcal{V} \right] \rho \quad (12)$$

by inserting $\alpha = \beta = 0$ in $\hat{\mathcal{L}}$ [41]. Taking the variational derivative with respect to ν and ρ , one obtains the following equations:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho \nabla\nu) = 0 \quad (13)$$

$$\frac{\partial\nu}{\partial t} = - \frac{1}{2} (\nabla\nu)^2 - w - \mathcal{V}. \quad (14)$$

The first of those equations is the continuity equation, while the second is Bernoulli's equation.

Going back to the quantum mechanical system described by equation (8), we introduce the following variable: $\hat{\nu} = \frac{\hbar\phi}{m} = \frac{S}{m}$. In terms of these new variables the Lagrangian density in equation (8) will take the form:

$$\mathcal{L} = - \left[\frac{\partial\hat{\nu}}{\partial t} + \frac{1}{2} (\nabla\hat{\nu})^2 + \frac{\hbar^2}{2m^2} \frac{(\nabla\sqrt{\hat{\rho}})^2}{\hat{\rho}} + \frac{1}{m} V \right] \hat{\rho} \quad (15)$$

When compared with equation (12) the following correspondence is noted:

$$\hat{\nu} \Leftrightarrow \nu, \quad \hat{\rho} \Leftrightarrow \rho, \quad \frac{\hbar^2}{2m^2} \frac{(\nabla\sqrt{\hat{\rho}})^2}{\hat{\rho}} \Leftrightarrow \varepsilon, \quad \frac{1}{m} V \Leftrightarrow \mathcal{V}. \quad (16)$$

The quantum "internal energy" $\frac{\hbar^2}{2m^2} \frac{(\nabla\sqrt{\hat{\rho}})^2}{\hat{\rho}}$ depends also on the derivative of the density and in this sense it is non local. This is unlike the fluid case, in which internal energy is a function of the mass density only. However, in both cases the internal energy is a positive quantity. We also notice that using the logarithmic variable $\lambda = \ln a$ we can write $\hat{\rho} = me^{2\lambda}$ and thus the quantum internal energy takes the simple form:

$$\varepsilon_q = \frac{\hbar^2}{2m^2} \frac{(\nabla\sqrt{\hat{\rho}})^2}{\hat{\rho}} = \frac{\hbar^2}{2m^2} (\nabla\lambda)^2. \quad (17)$$

In this case the Lagrangian density given in equation (15) takes the form:

$$\mathcal{L} = -me^{2\lambda} \left[\frac{\partial\hat{\nu}}{\partial t} + \frac{1}{2} (\nabla\hat{\nu})^2 + \frac{\hbar^2}{2m^2} (\nabla\lambda)^2 + \frac{1}{m} V \right] \quad (18)$$

Unlike classical systems in which the Lagrangian is quadratic in the time derivatives of the degrees of freedom, the Lagrangians of both quantum and fluid dynamics are linear in the time derivatives of the degrees of freedom.

Finally we note that the concept of quantum internal energy is closely related to its variational derivative the concept of quantum potential [3] (see the right side of equation (10)):

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2\sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}}. \quad (19)$$

And also that in the limit $\hbar \rightarrow 0$ Schrödinger's quantum mechanics is essentially a potential fluid flow without pressure or internal energy.

3.3 Madelung flows in terms of Fisher information

As explained in the introduction the quantum Madelung flow does not have a microstructure that will explain its internal energy. To understand the origins of this term let us look at the internal energy of equation (17), the term appearing in the Lagrangian density has the form:

$$\hat{\rho}\varepsilon_q = \frac{\hbar^2}{2m^2} (\nabla\sqrt{\hat{\rho}})^2 = \frac{\hbar^2}{2m} (\nabla a)^2. \quad (20)$$

Comparing this to equation (4) we arrive at the result:

$$\hat{\rho}\varepsilon_q = \frac{\hbar^2}{8m} \mathcal{F}_{Iq}. \quad \mathcal{F}_{Iq} \equiv 4(\nabla a)^2 \quad (21)$$

Thus the Lagrangian density of the Madelung flow can be written as:

$$\mathcal{L} = -\left[\frac{\partial\hat{\nu}}{\partial t} + \frac{1}{2} (\nabla\hat{\nu})^2 + \frac{1}{m} V \right] \hat{\rho} - \frac{\hbar^2}{8m} \mathcal{F}_{Iq} \quad (22)$$

The pre-factor $\frac{\hbar^2}{8m}$ seems to appear in every case in which Fisher information appears in a quantum Lagrangian and may be significant. This is the case also in spin fluid dynamics as will be shown in the next section.

4 Spin

Schrödinger's quantum mechanics is limited to the description of spin less particles and its fluid dynamics representation is limited to zero vorticity (potential) flows. This suggests that a quantum theory of particles with spin may have a fluid dynamics representation which is not limited to zero vorticity and thus requires the full Clebsch apparatus. The Pauli equation for a non-relativistic particle with spin is given by:

$$i\hbar\dot{\psi} = \hat{H}\psi, \quad \hat{H} = -\frac{\hbar^2}{2m}[\nabla - \frac{ie}{\hbar c}\mathbf{A}]^2 + \mu\mathbf{B} \cdot \boldsymbol{\sigma} + eA_0 + V \quad (23)$$

ψ here is a two dimensional complex column vector (also denoted as spinor), \hat{H} is a two dimensional hermitian operator matrix, e and μ are the charge and magnetic moment of the particle, c is the velocity of light in vacuum. The electromagnetic interaction is described by the vector \mathbf{A} and scalar A_0 potentials and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. $\boldsymbol{\sigma}$ is a vector of two dimensional Pauli matrices which can be represented as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (24)$$

A spinor ψ satisfying equation (23) must also satisfy a continuity equation of the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (25)$$

In the above:

$$\rho = \psi^\dagger \psi, \quad \mathbf{j} = \frac{\hbar}{2mi}[\psi^\dagger \nabla \psi - (\nabla \psi^\dagger)\psi] - \frac{e}{mc}\mathbf{A}\rho. \quad (26)$$

The symbol ψ^\dagger represents a row spinor (the transpose) whose components are equal to the complex conjugate of the column spinor ψ . Comparing the standard continuity equation to equation (25) suggests the definition of a velocity field as follows [3]:

$$\mathbf{v} = \frac{\mathbf{j}}{\rho} = \frac{\hbar}{2mi\rho}[\psi^\dagger \nabla \psi - (\nabla \psi^\dagger)\psi] - \frac{e}{mc}\mathbf{A}. \quad (27)$$

A variational description of the Pauli system can be given using the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}i\hbar(\dot{\psi}^\dagger \psi - \psi^\dagger \dot{\psi}) - \psi^\dagger \hat{H}\psi \quad (28)$$

Holland [3] has suggested the following representation of the spinor:

$$\psi = Re^{i\frac{\chi}{2}} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{i\frac{\phi}{2}} \\ i \sin\left(\frac{\theta}{2}\right) e^{-i\frac{\phi}{2}} \end{pmatrix} \equiv \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}. \quad (29)$$

In terms of this representation the density is given as:

$$\rho = \psi^\dagger \psi = R^2 \Rightarrow R = \sqrt{\rho}. \quad (30)$$

The mass density is given as:

$$\hat{\rho} = m\rho = m\psi^\dagger \psi = mR^2. \quad (31)$$

The probability amplitudes for spin up and spin down electrons are given by:

$$a_\uparrow = |\psi_\uparrow| = R \left| \cos \frac{\theta}{2} \right|, \quad a_\downarrow = |\psi_\downarrow| = R \left| \sin \frac{\theta}{2} \right| \quad (32)$$

Let us now look at the expectation value of the spin:

$$\langle \frac{\hbar}{2} \boldsymbol{\sigma} \rangle = \frac{\hbar}{2} \int \psi^\dagger \boldsymbol{\sigma} \psi d^3x = \frac{\hbar}{2} \int \left(\frac{\psi^\dagger \boldsymbol{\sigma} \psi}{\rho} \right) \rho d^3x \quad (33)$$

The spin density can be calculated using the representation given in equation (29) as:

$$\hat{s} \equiv \frac{\psi^\dagger \boldsymbol{\sigma} \psi}{\rho} = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta) \quad (34)$$

This gives an easy physical interpretation to the variables θ, ϕ as angles which describe the projection of the spin density on the axes. θ is the elevation angle of the spin density vector and ϕ is the azimuthal angle of the same. The velocity field can now be calculated by inserting ψ given in equation (29) into equation (27):

$$\mathbf{v} = \frac{\hbar}{2m} (\nabla \chi + \cos \theta \nabla \phi) - \frac{e}{mc} \mathbf{A}. \quad (35)$$

From now on we shall nullify the electromagnetic interaction for simplicity, hence:

$$\mathbf{v} = \frac{\hbar}{2m} (\nabla \chi + \cos \theta \nabla \phi). \quad (36)$$

Comparing equation (36) with the Clebsch form [41] suggest the following identification:

$$\alpha = \cos \theta, \quad \beta = \frac{\hbar}{2m} \phi, \quad \nu = \frac{\hbar}{2m} \chi. \quad (37)$$

Notice that α is single valued, but β and ν are not. Obviously this velocity field will have a generically non vanishing vorticity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \nabla \alpha \times \nabla \beta = \frac{\hbar}{2m} \nabla \cos \theta \times \nabla \phi = \frac{\hbar}{2m} \sin \theta \nabla \phi \times \nabla \theta \quad (38)$$

If we choose a local coordinate system R, θ, ϕ it is obvious that the spin vorticity will be always perpendicular to both the $\nabla \phi$ and $\nabla \theta$ directions (excluding the $\theta = n\pi$ case in which n is an integer). This means that $\boldsymbol{\omega}$ lies in both the ϕ and θ surfaces, i.e. the intersection of those surfaces. Hence it must have a component in the ∇R direction. If R, θ, ϕ would be standard spherical coordinates this would mean that the spin vorticity lies in the same direction as the spin density given in equation (34) but this will not be true in the general case.

Inserting the representation of ψ given in equation (29) into the Lagrangian density equation (28) will yield after tedious but straight forward calculations the Lagrangian density:

$$\begin{aligned}
 \mathcal{L}_P &\equiv -\hat{\rho}\left[\frac{\partial\nu}{\partial t} + \alpha\frac{\partial\beta}{\partial t} + \varepsilon_{qt} + \frac{V}{m} + \frac{1}{2}(\nabla\nu + \alpha\nabla\beta)^2\right] \\
 \varepsilon_{qt}[\hat{\rho}, \alpha, \beta] &\equiv \varepsilon_q[\hat{\rho}] + \varepsilon_{qs}[\alpha, \beta] \\
 \varepsilon_q[\hat{\rho}] &\equiv \frac{\hbar^2}{2m^2} \frac{(\nabla\sqrt{\hat{\rho}})^2}{\hat{\rho}} = \frac{\hbar^2}{2m^2} \frac{(\nabla R)^2}{R^2} \\
 \varepsilon_{qs}[\alpha, \beta] &\equiv \frac{\hbar^2}{8m^2} ((\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2) \\
 &= \frac{1}{2} \left(\left(\frac{\hbar}{2m}\right)^2 \frac{(\nabla\alpha)^2}{1-\alpha^2} + (1-\alpha^2)(\nabla\beta)^2 \right) \quad (39)
 \end{aligned}$$

The Lagrangian \mathcal{L}_P has the same form as the Clebsch Lagrangian $\hat{\mathcal{L}}$ given in [41]. However, there are some important differences. The internal energy in the Pauli Lagrangian is positive as for the barotropic fluid but now the internal energy depends on the derivatives of the degrees of freedom and not just on the density at given point in this sense this internal energy is non local. Moreover, it is made of two part the Schrödinger quantum internal energy ε_q which depends on the mass density and the spin quantum internal energy ε_{qs} that depend on the spin (vorticity) degrees of freedom. Finally the classical limit $\hbar \rightarrow 0$ will eliminate ε_q but will not eliminate the spin internal energy:

$$\lim_{\hbar \rightarrow 0} \varepsilon_{qs} = \frac{1}{2}(1-\alpha^2)(\nabla\beta)^2 \quad (40)$$

In this sense the Pauli theory has no standard classical limit, although this limit is a perfectly legitimate classical field theory. Taking the variational derivative we arrive at the equations of motion:

$$\begin{aligned}
 \frac{\partial\hat{\rho}}{\partial t} + \nabla \cdot (\hat{\rho}\mathbf{v}) &= 0 \\
 \frac{d\alpha}{dt} &= \frac{1}{\hat{\rho}} \nabla \cdot (\hat{\rho}(\alpha^2 - 1)\nabla\beta) \\
 \frac{d\beta}{dt} &= \left(\frac{\hbar}{2m}\right)^2 \frac{1}{\hat{\rho}\sqrt{1-\alpha^2}} \nabla \cdot \left(\hat{\rho} \frac{\nabla\alpha}{\sqrt{1-\alpha^2}}\right) + \alpha(\nabla\beta)^2 \\
 \frac{d\nu}{dt} &= \frac{1}{2}\mathbf{v}^2 - \mathcal{V} - \alpha^2(\nabla\beta)^2 - \frac{Q}{m} - \varepsilon_{qs} \\
 &\quad - \left(\frac{\hbar}{2m}\right)^2 \frac{\alpha}{\hat{\rho}\sqrt{1-\alpha^2}} \nabla \cdot \left(\hat{\rho} \frac{\nabla\alpha}{\sqrt{1-\alpha^2}}\right). \quad (41)
 \end{aligned}$$

We notice that in spin fluid dynamics α and β are not comoving scalar fields (labels) as in the case of ideal barotropic fluid dynamics. We also notice that external forces are only manifested through the ν equation through $\mathcal{V} = \frac{V}{m}$ as is the case in ideal barotropic fluid dynamics, although spin fluid dynamics contain additional quantum corrections. We are now in a position to calculate

the material derivative of the velocity and obtain the spin fluid dynamics Euler equation:

$$\frac{d\mathbf{v}}{dt} = -\nabla\left(\mathcal{V} + \frac{Q}{m}\right) - \left(\frac{\hbar}{2m}\right)^2 \frac{1}{\hat{\rho}} \partial_k(\hat{\rho} \nabla \hat{s}_j \partial_k \hat{s}_j) \quad (42)$$

Taking the curl of this equation we arrive at:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) - \left(\frac{\hbar}{2m}\right)^2 \nabla \times \left[\frac{1}{\hat{\rho}} \partial_k(\hat{\rho} \nabla \hat{s}_j \partial_k \hat{s}_j) \right] \quad (43)$$

Hence in spin fluid dynamics the vortex line do not move with the fluid as is the case of ideal barotropic flows. In this respect spin fluid dynamics is more reminiscent of non-barotropic flows in which the entropy gradient prevents the vortex line from co-moving. This result is in accordance with the equations of α and β , those fields are not comoving in spin fluid dynamics and neither is the intersection of their surfaces which correspond to the vorticity. Hence although the common Clebsch representation of spin and barotropic flows the dynamics is quite different and the differences do not vanish in the classical limit. According to equation (41) the discontinuities of both ν and β are comoving:

$$\frac{d[\nu]}{dt} = \frac{d[\beta]}{dt} = 0 \quad (44)$$

However as the vorticity lines are not comoving only the helicity per vortex flux is conserved but not the total helicity (see [41]).

4.1 Spin flows in terms of Fisher information

As explained in the introduction the quantum Spin flow does not have a microstructure that will explain its internal energy. To understand the origins of this term let us look at the internal energy of equation (39), the term appearing in the Lagrangian density has the form:

$$\hat{\rho} \varepsilon_{qt} = \hat{\rho} \varepsilon_q + \hat{\rho} \varepsilon_{qs} = \frac{\hbar^2}{8m} [4(\nabla R)^2 + R^2(\nabla \theta)^2 + R^2 \sin^2 \theta (\nabla \phi)^2] \quad (45)$$

Using the amplitudes of equation (32) and the definition of Fisher information density of equation (4) we arrive at the result:

$$\mathcal{F}_{I_p} = \mathcal{F}_{I_\uparrow} + \mathcal{F}_{I_\downarrow} = 4 [(\nabla a_\uparrow)^2 + (\nabla a_\downarrow)^2] = 4(\nabla R)^2 + R^2(\nabla \theta)^2 \quad (46)$$

Hence:

$$\hat{\rho} \varepsilon_{qt} = \frac{\hbar^2}{8m} \mathcal{F}_{I_p} + \frac{1}{2} \hat{\rho} (1 - \alpha^2) (\nabla \beta)^2 \quad (47)$$

Thus the Lagrangian density of the spin flow given in equation (39) can be written as:

$$\mathcal{L}_P = -\hat{\rho} \left[\frac{\partial \nu}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + \frac{1}{2} (1 - \alpha^2) (\nabla \beta)^2 + \frac{V}{m} + \frac{1}{2} (\nabla \nu + \alpha \nabla \beta)^2 \right] - \frac{\hbar^2}{8m} \mathcal{F}_{I_p} \quad (48)$$

The pre-factor $\frac{\hbar^2}{8m}$ seems to appear in every case in which Fisher information appears in a quantum Lagrangian and may be significant.

4.2 Towards a More complete description in terms of Fisher information

The definition of the velocity field given in equation (27) is not unique [3]. This definition is based on the conserved current given in equation (26). However it is clear that the current:

$$\mathbf{j}_{total} = \mathbf{j} + \nabla \times \mathbf{G} \quad (49)$$

is also conserved for an arbitrary \mathbf{G} . This will lead to a new fluid dynamics with a different velocity:

$$\mathbf{v}_{total} = \mathbf{v} + \frac{1}{\hat{\rho}} \nabla \times \mathbf{G}. \quad (50)$$

For instance Holland [3] has suggested to consider the form:

$$\mathbf{v}_{total} = \mathbf{v} + \frac{\hbar}{2m\hat{\rho}} \nabla \times (\hat{\rho}\hat{\mathbf{s}}). \quad (51)$$

Demanding that α is comoving:

$$\frac{\partial \alpha}{\partial t} + \mathbf{v}_{total} \cdot \nabla \alpha = 0 \quad (52)$$

and using equation (50) and equation (41) will result in:

$$\frac{1}{\hat{\rho}} \nabla \times \mathbf{G} \cdot \nabla \alpha = \frac{1}{\hat{\rho}} \nabla \cdot (\hat{\rho}(1 - \alpha^2) \nabla \beta) \quad (53)$$

Or:

$$\nabla \times \mathbf{G} \cdot \nabla \alpha = \nabla \cdot (\alpha \nabla \times \mathbf{G}) = \nabla \cdot (\hat{\rho} \sin^2 \theta \nabla \beta) \quad (54)$$

Thus a comoving velocity will exist provided there is a \mathbf{A} such that:

$$\frac{1}{\hat{\rho}} \nabla \times \mathbf{G} = \frac{\sin^2 \theta}{\cos \theta} \nabla \beta + \frac{1}{\cos \theta \hat{\rho}} \nabla \times \mathbf{A} \quad (55)$$

It will be interesting to see if one can define a fluid dynamics such that vortex lines are comoving. And perhaps when such velocity field is found one will be able to obtain a more accurate correspondence between Fisher information and internal energy.

5 Conclusion

In this paper the original two state system was revisited, that is Pauli's electron with a spin. It was shown how Pauli's theory can be formulated as a spin fluid dynamics in terms of a Clebsch representation. The theory is given in terms of a variational principle and the fluid equations are derived. The similarities and differences with barotropic fluid dynamics were discussed. Although the theories have similar Lagrangian densities it is shown that the α and β variables

are not comoving in spin fluid dynamics nor do the vortex lines move with the flow. This means that the topological invariants connected to vortex motion which exist in barotropic flows are not invariant in spin fluid dynamics.

A fundamental problem in the fluid mechanical interpretation of quantum mechanics still exist. This refers to the meaning of thermodynamic quantities which are part of fluid mechanics. In thermodynamics Concepts like specific enthalpy, pressure and temperature are derivatives of the specific internal energy which is given in terms of the equation of state as function of entropy and density. The internal energy is a part of any Lagrangian density attempting to describe fluid dynamics. The form of the internal energy can in principle be explained on the basis of the microscopic composition of the fluid, that is the atoms and molecules from which the fluid is composed and their interactions using statistical mechanics. However, the quantum fluid has no microscopic structure and yet analysis of the equations of both the spin less [5,6] and spin [22] quantum fluid dynamics shows that terms analogue to internal energies appear in both cases. The question then arises where do those internal energies come from, surely one would not suggest that the quantum fluid has a microscopic sub structure as this will defy the conception of the electron as a fundamental particle. The answer to this question comes from an entirely different discipline of measurement theory [38]. Fisher information a basic notion of measurement theory is a measure of the quality of the measurement of any quantity. It was shown that this concept is proportional to the internal energy of a spin less electron and can explain most parts of the internal energy of an electron with spin.

To conclude we suggest the following future directions of research:

1. It is conjectured that same analogy found between Pauli's theory and fluid dynamics may be found between Dirac's relativistic electron theory and relativistic fluid dynamics.
2. Electromagnetic fields were nulled in the present paper. It will be interesting to study the connections between the theory of a charged fluid and the full Pauli theory which includes electromagnetic interactions.

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